Part VI
An introduction to the theory of zeta-functions of root systems

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1 Introduction

The theory of multiple zeta-functions has a long history, from the work of Barnes and Mellin at the beginning of the 20th century, or even from the days of Euler. A new stream of research of multiple zeta-functions began in 1990s, when some fascinating connections between the theory of multiple zeta-functions and various branches of mathematics and mathematical physics were discovered. An epoch-making paper is Zagier [33], in which two types of multiple zeta-functions are discussed. One is the r-fold zeta-function of the form

$$\begin{aligned}\zeta_{EZ,r}(s_1,\ldots,s_r) = \sum_{m_1=1}^{\infty}\cdots\sum_{m_r=1}^{\infty} & m_1^{-s_1}(m_1+m_2)^{-s_2} \\ & \times\cdots\times(m_1+\cdots+m_r)^{-s_r},\end{aligned} \tag{1.1}$$

which is now sometimes called the Euler-Zagier r-ple zeta-function. Zagier [33] considered the values of (1.1) when $s_1,\ldots,s_r$ are positive integers and $s_r \geq 2$. Note that Hoffman [5] independently studied the same values at about the same time.

Another type of multiple zeta-functions discussed in Zagier's paper is the class of Witten's zeta-functions. Let $\mathfrak{g}$ be a complex semisimple Lie algebra. The Witten zeta-function associated with $\mathfrak{g}$ is defined as

$$\zeta_W(s;\mathfrak{g}) = \sum_{\varphi}(\dim\varphi)^{-s}, \tag{1.2}$$

where φ runs over all finite dimensional irreducible representations of $\mathfrak{g}$. Special values of this series were first studied by Witten [32] in connection with a problem in quantum gauge theory. As we will see later, we can write down a more explicit form of $\zeta_W(s;\mathfrak{g})$ by using Weyl's dimension formula. We will find that the explicit form of $\zeta_W(s;\mathfrak{g})$ is an r-fold sum, where r is the rank of $\mathfrak{g}$. Therefore $\zeta_W(s;\mathfrak{g})$ is a kind of multiple zeta-functions.

The original form of $\zeta_W(s;\mathfrak{g})$ is a function in one variable, though the sum in the definition is multiple. However it has been noticed recently that, for deeper investigations of $\zeta_W(s;\mathfrak{g})$, it is convenient to introduce the multi-variable generalization of $\zeta_W(s;\mathfrak{g})$ and discuss its properties. This is the main theme of the present article.

Since the present lecture is of introductory nature, we begin with the explanation of the basic theory of Lie algebras. It is impossible to give the full account of the theory here; for the details, see, for example, [2], [7], or [20].

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