Part 3A. The canonical S-operator and the canonical class of linear differential equations of second order on algebraic function field L of one variable over $\mathbf{C}$, and their algebraic characterizations when L is "arithmetic".

The S-operators.

§37. The symbol $\langle \eta, \xi \rangle$. Let L be any field, let $D(L)$ be a *one*-dimensional vector space over L, and let $d : L \to D(L)$ be a map satisfying $d(x+y) = dx + dy$, $d(xy) = xdy + ydx$ for all $x, y \in L$. For each positive integer h, denote by $D^h(L)$ the tensor product $D(L) \otimes \cdots \otimes D(L)$ (h copies) over L (so that $\dim_L D^h(L) = 1$), and call the elements of $D^h(L)$ *differentials of degree h* (in L). Put $D(L)^\times = D(L) \setminus \{0\}$. Then if ξ is any fixed element of $D(L)^\times$, the elements of $D^h(L)$ are expressed uniquely in the form $a \cdot \xi^h$ ($a \in L$). Here, ξ^h will always denote $\xi \otimes \cdots \otimes \xi$ (h copies). For any $\xi \in D(L)^\times$ and $\eta \in D(L)$, the number $a \in L$ with $\eta = a\xi$ will be denoted by η/ξ. Finally, we shall denote by k the constant field, i.e., $k = \{x \in L \mid dx = 0\}$. It is clear that k is a subfield of L.

Now for each $\xi, \eta \in D(L)^\times$, an element $\langle \eta, \xi \rangle$ of $D^2(L)$ is defined in the following way. Put $w_1 = \eta/\xi$, $w_{i+1} = dw_i/\xi$ ($i \geq 1$). Then

Definition.

$$\langle \eta, \xi \rangle = \frac{2w_1w_3 - 3w_2^2}{w_1^2}\,\xi^2.$$

In particular, if $x, y \in L\backslash k$, then we have

$$\langle dy, dx \rangle = \frac{2\left(\frac{dy}{dx}\right)\left(\frac{d^3y}{dx^3}\right) - 3\left(\frac{d^2y}{dx^2}\right)^2}{\left(\frac{dy}{dx}\right)^2}(dx)^2, \tag{76}$$

where $\frac{d^i}{dx^i} = \left(\frac{d}{dx}\right)^i$ ($i \geq 1$). Thus $\langle \eta, \xi \rangle$ is, so to speak, the "algebraic Schwarzian derivative". The following Proposition is classically well-known for the analytic Schwarzian derivative.

Proposition 7. (i) *For any $\xi, \eta, \zeta \in D(L)^\times$, we have*

$$\langle \eta, \zeta \rangle - \langle \xi, \zeta \rangle = \langle \eta, \xi \rangle. \tag{77}$$

(ii) *Let $\eta \in D(L)^\times$ and $x \in L\backslash k$. Then $\langle \eta, dx \rangle = 0$ if and only if η is of the form $\eta = dx_1$ with $x_1 = \frac{ax+b}{cx+d}$, $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in GL_2(k)$.*[20]

[20] Here, the same notation d is used for the map $d : L \to D(L)$ and for the $(2,2)$-element of the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$. I hope that this will not confuse the readers.