LECTURE NOTES ON GEOMETRIC CRYSTALS AND THEIR COMBINATORIAL ANALOGUES

ARKADY BERENSTEIN AND DAVID KAZHDAN

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1. FIRST LECTURE: GEOMETRIC AND UNIPOTENT CRYSTALS

The first author would like to express his gratitude to Professor Okado for the opportunity to give the lectures, to Professor Kuniba for the great help in preparation of the lecture notes, and to RIMS for the hospitality.

The lectures are based on the results of our research originated six years ago in [1] and continued in [2]. We start with the problem of birational Weyl group actions that served as the original motivation for this work (for the terminology and results on reductive algebraic groups, see, e.g., [8]).

Problem 1.1. Let G be a split reductive algebraic group with a maximal torus T. Given an affine algebraic variety X, a function f on X, and a morphism of algebraic varieties $\gamma : X \to T$, construct a birational action of the Weyl group $W = Norm_T(G)/T$ on X in such a way that:

(1) The structure map $\gamma : X \to T$ commutes with the W-action (where the W-action on T is the natural one).
(2) The function f is W-invariant.
(3) For each $w \in W$ the fixed point set $X^w = \{x \in X : w(x) = x\}$ is the pre-image $\gamma^{-1}(T^w)$ of the fixed point set $T^w = \{t \in T : w(t) = t\}$ (i.e., all fixed points of w "upstairs" come from the fixed points of w "downstairs").

Each solution of the problem defines a version of a *W-equivariant algebro-geometric distribution* Φ_T on T from [3, Section 7.10] (the above condition (3) serves as a natural analogue of the requirement 3) from [3, Section 7.10]).

Conjecture 7.11 from [3] asserts that for each algebraic ℓ-dimensional representation ρ of the *Langlands dual group* $G^\vee$ there exists a W-equivariant algebro-geometric distribution $\Phi_{\rho,T}$ with $X = \mathbb{G}_m^\ell$ (where $\mathbb{G}_m$ stands for the multiplicative group), $f(c_1, \ldots, c_\ell) = \sum_{i=1}^{\ell} c_i$, and $\gamma = \gamma_\rho : X \to T$ is the homomorphism of algebraic tori determined by ρ. The same conjecture claims that the existence of such $\Phi_{\rho,T}$ implies a corollary from Local Langlands conjectures (see [3, Section 1.1] for details).

Therefore, solving Problem 1.1 will help to dealing with the Local Langlands conjectures.

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