4. Fundamental groups

In this section we give a review of some background material relating to fundamental groups and covering spaces. Since these serve primarily as illustrative examples, our presentation will be informal, and we omit proofs. We will describe some consequences for free groups.

4.1. Definition of fundamental groups.

The fundamental group is an invariant of a topological space. We are not interested here in "pathological" examples. We will generally assume that our spaces are reasonably "nice". For example, manifolds and simplicial complexes are all "nice", and in practice that is all we care about.

Let X be a topological space. Fix a "basepoint" $p \in X$. A *loop* based at p is a path $\alpha : [0,1] \longrightarrow X$ with $\alpha(0) = \alpha(1) = p$. Two such loops, α, β, are *homotopic* if one can be deformed to the other through other loops; more precisely, if there is a map $F : [0,1]^2 \longrightarrow X$ with $F(t,0) = \alpha(t)$, $F(t,1) = \beta(t)$ and $F(0,u) = F(1,u) = p$ for all $t, u \in [0,1]$. This defines an equivalence relation on the set of paths. Write $[\alpha]$ for the homotopy class of α.

Given loops α, β, write $\alpha * \beta$ for the path that goes around α (twice as fast) then around β (i.e. $\alpha * \beta(t)$ is $\alpha(2t)$ for $t \leq 1/2$ and $\beta(2t-1)$ for $t \geq 1/2$) (Figure 4a). Write $[\alpha][\beta] = [\alpha * \beta]$.

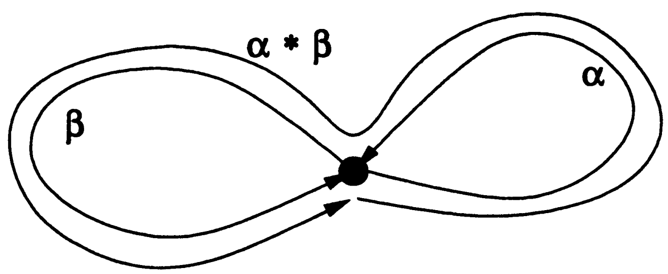

Figure 4a.