3 Comparison principle

In this section, we discuss the comparison principle, which implies the uniqueness of viscosity solutions when their values on $\partial\Omega$ coincide (*i.e.* under the Dirichlet boundary condition). In the study of the viscosity solution theory, the comparison principle has been the main issue because the uniqueness of viscosity solutions is harder to prove than existence and stability of them.

First, we recall some "classical" comparison principles and then, show how to modify the proof to a modern "viscosity" version.

In this section, the comparison principle roughly means that

"Comparison principle"

$$\boxed{\left.\begin{array}{c}\textbf{viscosity subsolution } u\\ \textbf{viscosity supersolution } v\\ u \le v \textbf{ on } \partial\Omega\end{array}\right\} \quad \Longrightarrow \quad u \le v \textbf{ in } \overline{\Omega}}$$

Modifying our proofs of comparison theorems below, we obtain a slightly stronger assertion than the above one:

$$\boxed{\left.\begin{array}{c}\text{viscosity subsolution } u\\ \text{viscosity supersolution } v\end{array}\right\} \quad \Longrightarrow \quad \max_{\overline{\Omega}}(u-v) = \max_{\partial\Omega}(u-v)}$$

We remark that the comparison principle implies the uniqueness of (continuous) viscosity solutions under the Dirichlet boundary condition:

"Uniqueness for the Dirichlet problem"

$$\boxed{\left.\begin{array}{c}\textbf{viscosity solutions } u \textbf{ and } v\\ u = v \textbf{ on } \partial\Omega\end{array}\right\} \quad \Longrightarrow \quad u = v \textbf{ in } \overline{\Omega}}$$

Proof of "the comparison principle implies the uniqueness".

Since u (resp., v) and v (resp., u), respectively, are a viscosity subsolution and supersolution, by $u = v$ on $\partial\Omega$, the comparison principle yields $u \le v$ (resp., $v \le u$) in $\overline{\Omega}$. □

In this section, we mainly deal with the following PDE instead of (2.6).

$$\nu u + F(x, Du, D^2u) = 0 \quad \text{in } \Omega, \tag{3.1}$$

where we suppose that

$$\nu \ge 0, \tag{3.2}$$

and

$$F : \Omega \times \mathbf{R}^n \times S^n \to \mathbf{R} \quad \text{is continuous.} \tag{3.3}$$