1 Introduction

Throughout this book, we will work in Ω (except in sections 4.2 and 5.4), where

$$\Omega \subset \mathbf{R}^n \text{ is open and bounded.}$$

We denote by $\langle \cdot, \cdot \rangle$ the standard inner product in $\mathbf{R}^n$, and set $|x| = \sqrt{\langle x, x \rangle}$ for $x \in \mathbf{R}^n$. We use the standard notion of open balls: For $r > 0$ and $x \in \mathbf{R}^n$,

$$B_r(x) := \{y \in \mathbf{R}^n \mid |x - y| < r\}, \quad \text{and} \quad B_r := B_r(0).$$

For a function $u : \Omega \to \mathbf{R}$, we denote its gradient and Hessian matrix at $x \in \Omega$, respectively, by

$$Du(x) := \begin{pmatrix} \frac{\partial u(x)}{\partial x_1} \\ \vdots \\ \frac{\partial u(x)}{\partial x_n} \end{pmatrix},$$

$$D^2u(x) := \begin{pmatrix} \frac{\partial^2 u(x)}{\partial x_1^2} & \cdots & j\text{-th} & \cdots & \frac{\partial^2 u(x)}{\partial x_1 \partial x_n} \\ \vdots & & \vdots & & \vdots \\ i\text{-th} & \cdots & \frac{\partial^2 u(x)}{\partial x_i \partial x_j} & \cdots & \vdots \\ \vdots & & \vdots & & \vdots \\ \frac{\partial^2 u(x)}{\partial x_n \partial x_1} & \cdots & \cdots & \cdots & \frac{\partial^2 u(x)}{\partial x_n^2} \end{pmatrix}.$$

Also, S^n denotes the set of all real-valued $n \times n$ symmetric matrices. Note that if $u \in C^2(\Omega)$, then $D^2u(x) \in S^n$ for $x \in \Omega$.

We recall the standard ordering in S^n:

$$X \leq Y \quad \Longleftrightarrow \quad \langle X\xi, \xi \rangle \leq \langle Y\xi, \xi \rangle \qquad \text{for } \forall \xi \in \mathbf{R}^n.$$

We will also use the following notion in sections 6 and 7: For $\xi =^t (\xi_1, \ldots, \xi_n)$, $\eta =^t (\eta_1, \ldots, \eta_n) \in \mathbf{R}^n$, we denote by $\xi \otimes \eta$ the $n \times n$ matrix whose (i, j)-entry is $\xi_i \eta_j$ for $1 \leq i, j \leq n$;

$$\xi \otimes \eta = \begin{pmatrix} \xi_1 \eta_1 & \cdots & j\text{-th} & \cdots & \xi_1 \eta_n \\ \vdots & & \vdots & & \vdots \\ i\text{-th} & \cdots & \xi_i \eta_j & \cdots & \vdots \\ \vdots & & \vdots & & \vdots \\ \xi_n \eta_1 & \cdots & \cdots & \cdots & \xi_n \eta_n \end{pmatrix}.$$