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The modular cocycle from commensuration and its Mackey range

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§1. Introduction

Let Γ be a group. A subgroup E of Γ is called *quasi-normal* (or *commensurated*) in Γ if for any $\gamma \in \Gamma$, the group $E \cap \gamma E \gamma^{-1}$ is of finite index in both E and $\gamma E \gamma^{-1}$. In this case, the *modular homomorphism* $\mathfrak{m}\colon \Gamma \to \mathbb{R}_+^\times$ into the multiplicative group $\mathbb{R}_+^\times$ of positive real numbers is defined by the formula

$$\mathfrak{m}(\gamma) = [E : E \cap \gamma E \gamma^{-1}][\gamma E \gamma^{-1} : E \cap \gamma E \gamma^{-1}]^{-1}$$

for $\gamma \in \Gamma$. This $\mathfrak{m}$ depends only on the commensurability class of E, where two subgroups of Γ are called *commensurable* if their intersection is of finite index in both of them. If that class is characteristic in Γ, then $\mathfrak{m}$ is invariant under any automorphism of Γ, and we can derive valuable information on Γ from $\mathfrak{m}$. With regard to the *Baumslag-Solitar* (*BS*) *group* defined by the presentation

$$\mathrm{BS}(p, q) = \langle\, a,\, t \mid t a^p t^{-1} = a^q \,\rangle,$$

where p and q are integers with $2 \leq p \leq |q|$, the modular homomorphism $\mathfrak{m}$ is associated to the quasi-normal subgroup $\langle a \rangle$, and it turns out that the image of $\mathfrak{m}$ and hence the ratio $|q/p|$ is an isomorphism invariant among the BS groups. In [Ki2, Theorem 1.2], we realized this for transformation-groupoids from the BS groups. Namely, to the pair of a discrete measured groupoid and its quasi-normal subgroupoid, the modular cocycle is associated, and among transformation-groupoids from the BS groups, its Mackey range is shown to be an isomorphism invariant of the groupoid. This work was inspired by construction of the

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