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Mapping configuration spaces to moduli spaces

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§1. Introduction

Let $\mathcal{C}_n$ be the space of unordered n-tuples of distinct points in the interior of the unit disc $D = \{z \in \mathbb{C} : |z| \leq 1\}$, and let $\mathcal{M}_{g,2}$ denote the moduli space of connected Riemann surfaces of genus g with two ordered and parametrised boundary components. There is a natural map

$$\Phi : \mathcal{C}_{2g+2} \longrightarrow \mathcal{M}_{g,2};$$

it takes a subset $\underline{a} = \{a_1, \ldots, a_{2g+2}\} \subset D$ to the part $\Sigma_{\underline{a}}$ of the Riemann surface associated to the function

$$f_{\underline{a}}(z) = ((z - a_1) \ldots (z - a_{2g+2}))^{1/2}$$

which lies over the disc D.

The purpose of this paper is to describe this map in topological terms. On passing to the fundamental groups it gives a geometric definition of a well-known algebraic homomorphism $\phi : \mathfrak{Br}_{2g+2} \to \Gamma_{g,2}$ from the braid group on $2g+2$ strings to the mapping class group which is described below. The main result is that Φ is compatible with naturally defined actions of the framed little 2-discs operad on configuration spaces and moduli spaces. As immediate consequences we deduce that Φ is trivial in homology with field coefficients, and trivial on stable homology with any constant coefficient system. In particular, this proves an unstable version of a conjecture by Harer, and simplifies the proof of the triviality of the map in stable homology given in [ST], to which we refer for more background on this problem.

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