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Polyhedral Algebras, Arrangements of Toric Varieties, and their Groups

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§1. Introduction

In our previous paper [BG] we generalized standard properties of the group $\mathrm{GL}_n(k)$ of graded automorphisms of the polynomial ring $k[x_1,\dots,x_n]$ over a field k to the group $\operatorname{gr.aut}(k[S_P])$ of graded automorphisms of a polytopal k-algebra $k[S_P]$ associated with a lattice polytope P. The generators of the k-algebra $k[S_P]$ correspond bijectively to the lattice points in P, and their relations are the binomials representing the affine dependencies of the lattice points. (See Bruns, Gubeladze, and Trung [BGT] for polytopal algebras.) Thus $k[x_1,\dots,x_n]$ can be viewed as the polytopal algebra $k[S_{\Delta_{n-1}}]$ for the unit $(n-1)$-simplex Δ_{n-1}, and the fact that every invertible matrix can be reduced to a diagonal one by elementary row transformations is then a special case of our theorem [BG, Theorem 3.2] that every element of $\operatorname{gr.aut}(k[S_P])$ is a composition of elementary automorphisms, toric automorphisms, and affine symmetries of the polytope. (The symmetries are only needed if $\operatorname{gr.aut}(k[S_P])$ is not connected.) Polytopal algebras and their normalizations are special instances of affine semigroup algebras; more generally, we have described the group of graded automorphisms of an arbitrary normal affine semigroup algebra [BG, Remark 3.3(c)].

In [BG] an application to toric geometry is a description of the automorphism group of a projective toric variety over an algebraically closed field of arbitrary characteristic. Our approach avoids the theory of linear algebraic groups, and for projective toric varieties we have strengthened the classical theorem of Demazure [De] and its recent generalizations by Cox [Co] and Bühler [Bu].

The main issue of this paper is a generalization from the case of a single polytope to algebras $k[\Pi]$ corresponding to *lattice polyhedral*

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