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On the Prime Graph of a Finite Simple Group An Application of the Method of Feit-Thompson-Bender-Glauberman

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Introduction The theorem alluded to in the subtitle is the Odd Order Theorem of Feit-Thompson [FT] which states that all finite groups of odd order are solvable. For the remarkable proof, they invented a revolutionary new method which was influential to the development of finite group theory in the last 30 odd years. Recently, Bender and Glauberman [BG] have published a highly polished proof covering the group theoretical portion of the proof of the Odd Order Theorem.

By design, their proof is by contradiction. From the start they work on the hypothetical minimal simple group of odd order and study its properties. Thus, all the wonderful intermediate results are properties of the hypothetical group, and hence they may be vacuous. One of the goals of this paper is to show that this is not so; their method does give positive results and all the intermediate results are in fact properties of some real groups.

We consider the prime graph $\Gamma(G)$ of a finite group G. This is the graph defined as follows. The set of vertices of $\Gamma(G)$ is the set $\pi(G)$ of the primes dividing the order $|G|$ of G. If $p, q \in \pi(G)$, we join p and q by an edge in $\Gamma(G)$ if and only if $p \neq q$ and G has an element of order pq.

The classification of finite simple groups has several interesting consequences on the prime graph of a finite group. The following is one of them.

Theorem A. *Let Δ be a connected component of the prime graph $\Gamma(G)$ of a finite group G, and let ϖ be the set of primes in Δ. Assume that $\Delta \neq \Gamma(G)$ and $2 \notin \varpi$. Then, Δ is a clique.*

Usually, we identify Δ with ϖ and abuse the terms, saying ϖ is a connected component of the graph $\Gamma(G)$. Theorem A has not been stated

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