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Iwasawa Theory – Past and Present

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Dedicated to the memory of Kenkichi Iwasawa

Let F be a finite extension of $\mathbb{Q}$. Let p be a prime number. Suppose that F_∞ is a Galois extension of F and that $\Gamma = \mathrm{Gal}(F_\infty/F)$ is isomorphic to $\mathbb{Z}_p$, the additive group of p-adic integers. The nontrivial closed subgroups of Γ are of the form $\Gamma_n = \Gamma^{p^n}$ for $n \geq 0$. They form a descending sequence and Γ/Γ_n is cyclic of order p^n. If we let $F_n = F_\infty^{\Gamma_n}$, then we obtain a tower of number fields

$$F = F_0 \subset F_1 \subset \cdots \subset F_n \subset \cdots$$

such that F_n/F is a cyclic extension of degree p^n and $F_\infty = \bigcup_n F_n$. In 1956, at the summer meeting of the American Mathematical Society in Seattle, Iwasawa gave an invited address entitled *A theorem on Abelian groups and its application to algebraic number theory.* The application which he discussed is the following now famous theorem.

Theorem. *Let p^{e_n} be the highest power of p dividing the class number of F_n. Then there exist integers λ, μ, and ν such that $e_n = \lambda n + \mu p^n + \nu$ for all sufficiently large n.*

Iwasawa's proof of this theorem is based on studying the Galois group $X = \mathrm{Gal}(L_\infty/F_\infty)$, where $L_\infty = \bigcup_n L_n$ and L_n is the p-Hilbert class field of F_n. (That is, L_n is the maximal abelian p-extension of F_n which is unramified at all primes of F_n. By class field theory, L_n is a finite extension of F_n and $[L_n : F_n] = p^{e_n}$.) The extension L_∞/F is Galoisian, and one has an exact sequence

$$0 \to X \to \mathrm{Gal}(L_\infty/F) \to \Gamma \to 0.$$

Since X is a projective limit of finite abelian p-groups, we can regard X as a compact $\mathbb{Z}_p$-module. ($\mathbb{Z}_p$ denotes the ring of p-adic integers.) But

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