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Hodge Cycles and Unramified Class Fields

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In the theory of complex multiplication, we obtain ramified class fields by the torsion points of a CM abelian variety and unramified class fields as certain fields of moduli (cf. [S-T]). Several authors studied the class fields obtained by complex multiplication ([K], [R], [S], [Mau], [O]).

When the abelian variety has "many" Hodge cycles (hence the Hodge group is "small"), it is known that the ramified class fields are "small" (cf. [R]). On the contrary, we know no clear relation between Hodge cycles and unramified class fields. In certain cases, however, the existence of exceptional Hodge cycles helps us to say something about the unramified class fields.

In the previous paper [DCM], we have shown a method of constructing CM abelian varieties with exceptional Hodge cycles and we have given several examples in which we can determine the degrees of the unramified class fields obtained as fields of moduli.

In the present note, we shall generalize the result of [DCM] and explain how the exceptional Hodge cycles influence the unramified class fields.

§1. Exceptional Hodge Cycles

Let A be a CM abelian variety of type (K, S) defined over a subfield of $\mathbf{C}$ with $\dim A = d$, where K is a CM-field of degree $2d$ and S is a CM-type of K (cf. [S-T]). Let $\mathbf{Hg} = \mathbf{Hg}(A)$ be the Hodge (or the special Mumford-Tate) group of A, that is a sub algebraic torus of $GL(H^1(A, \mathbf{Q}))$. It is known that $\dim \mathbf{Hg} \leq d$. When A is simple and $\dim \mathbf{Hg} < d$, there exist certain Hodge cycles on a product A^k of several copies of A that are not generated by the divisor classes; we call such Hodge cycles *exceptional* (cf. [DCM]).

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