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Galois Module Structure of p-Class Formations

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0. Introduction

One of the main subjects in (classical) class field theory is to study the structure of the Galois groups $\mathcal{A}_L$ of abelian extensions of a local or global field L. One natural next step would be to take a Galois extension L/K with given group G and to investigate the structure of $\mathcal{A}_L$ as a G-module. This has been done by several authors (see e.g. [J_2], [N_2],..., and the references therein), mainly from the p-adic point of view: they single out a prime number p and focus their investigation on the $\mathbb{Z}_p[G]$-module structure of the p-Sylow subgroup A_L of $\mathcal{A}_L$. In the most interesting cases, it happens that (for a fixed base field K), the modules A_L constitute a so-called "p-class formation" (see §1); so the next natural step is to replace the A_L by the modules X_L belonging to any p-class formation. Adding noetherian conditions, we obtain (Thm. 3.2 (below)) that, up to projective summands, the $\mathbb{Z}_p[G]$-module X_L is determined by its $\mathbb{Z}_p$-torsion tX_L and a certain character χ_L of the group $H^2(G, tX_L)$. This generalizes a former result of U. Jannsen on the "homotopy type" of A_L ([J_2], Thm. 4.5) and could probably be proved by extending the methods of [J_2]. In order to throw some new light on the problem, we preferred instead to employ the technique of "envelopes" introduced by Gruenberg and Weiss ([GW], [W]) in their study of the Stark conjecture.

This paper (the first part of which is semi-expository) will be organized as follows: after recalling some known facts on p-class formations (§1) and the homotopy of modules (§2), we prove the main theorem in §3, essentially by giving a canonical description of the envelope of X_L by means of a relative Weil group, and of the character χ_L by means of a "trace form". As an illustration, we study in §4 the arithmetic of

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