Advanced Studies in Pure Mathematics 24, 1996
Progress in Algebraic Combinatorics
pp. 1–80

Combinatorial Cell Complexes

Michael Aschbacher

We define and discuss a category of combinatorial objects we call *combinatorial cell complexes* and a functor T from this category to the category of topological spaces with cell structure, whose image is closely related to the category of CW-complexes. This formalism was developed to study finite group actions on topological spaces. In order to make effective use of our detailed knowledge of the finite simple groups, it seems necessary to make such a translation from a purely topological setting to the language of geometric combinatorics.

Our functor T assigns to each combinatorial cell complex X its geometric realization $T(X)$. We show the functor T defines an equivalence of categories between the category of combinatorial cell complexes whose cell boundaries are spheres, and a certain subcategory of CW-complexes we call normal CW-complexes.

We often concentrate on a subcategory of combinatorial cell complexes we call restricted combinatorial cell complexes; the restricted CW-complexes are the CW-complexes corresponding to the restricted combinatorial cell complexes under our equivalence of categories. Restricted CW-complexes include regular CW-complexes but also many other classical examples like the torus, the Klein bottle, and the Poincaré dodecahedron, which are discussed here as illustrations.

We associate to each restricted combinatorial cell complex X, a simplicial complex $K(X)$ and a canonical triangulation of $T(X)$ by $K(X)$. The geometric realization of a general combinatorial cell complex can also be canonically triangulated, but by a more complicated simplicial complex than $K(X)$. However we do not supply a proof of this last fact here.

We define cellular homology combinatorially, and show that if X is restricted and the boundary of each cell is homologically spherical, then

---

Received November 3, 1994.
Revised February 22, 1995.
This work was partially supported by NSF DMS–9101237 and BSF 92-00320