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Trudinger's Inequality and Related Nonlinear Elliptic Equations in Two-Dimension

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§1. Introduction and results

We are concerned with the following nonlinear elliptic equations:

$$\begin{cases} -\Delta u = \lambda u e^{u^2}, & x \in B, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{1}$$

where $B = B_1(0) \subset \mathbb{R}^2$ is a unit disk in $\mathbb{R}^2$ and λ is a positive parameter. We consider a family of solutions of (1) satisfying

$$\|u\|_{L^\infty} \to \infty \quad \text{as } \lambda \to 0. \tag{2}$$

The nonlinearity of the equation (1) is the Sobolev critical exponent in two-dimension. For any domain $\Omega \in \mathbb{R}^2$, It is well known that the Sobolev space $H_0^1(\Omega)$ is continuously imbedded in $L^p(\Omega)$ for any $p < \infty$ but is false in the case $p = \infty$. Trudinger [18] showed that for any $u \in H_0^1(\Omega)$ with $\|\nabla u\|_2 = 1$, there are two constants $\alpha > 0$ and $C > 0$ such that

$$\int_\Omega \exp\{\alpha u^2\} dx \leq C|\Omega|. \tag{3}$$

Later, Moser [7] simplified the proof and improved that (3) is also valid for $\alpha \leq 4\pi$. Here 4π is the constant of the isoperimetric inequality. The inequality (3) is also valid for any unbounded domain (Ogawa [9]). That is when Ω is any domain in $\mathbb{R}^2$, we have for all $u \in H_0^1(\Omega)$,

$$\int_\Omega \{\exp(u^2) - 1\} dx \leq C\|u\|_2^2, \quad \|\nabla u\|_2 = 1. \tag{4}$$

(See also Ogawa-Ozawa [10] and Ozawa [12] for further extensions).

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