Advanced Studies in Pure Mathematics 23, 1994
Spectral and Scattering Theory and Applications
pp. 239–246

Helmholtz-Type Equation on Non-compact Two-Dimensional Riemannian Manifolds

Reiji Konno

§1. Introduction

We shall consider the existence, or rather non-existence of square integrable solutions of the equation $-\triangle f = \lambda f$ on a non-compact Riemannian manifold which is homeomorphic to $\mathbf{R}^n$ minus a ball, where $\triangle$ is the Laplace-Beltrami operator and λ is an arbitrary positive constant. The source of this problem is the study of the non-existence of positive eigenvalues of the Schrödinger operator $-\triangle + q$ in a region of $\mathbf{R}^n$, and the method used there was found to be applicable to problems of the above type.

There may be several ways of physical interpretation of the equation $-\triangle f = \lambda f$ on manifolds. But probably the most essential one is as follows: Let a Riemannian manifold $\mathcal{M}$ represent a non-Euclidean space which is filled up with a medium whose displacement on some quantity, e.g. pressure, electric field etc., obeys Hooke's law isotropically and homogeneously in each small portion of the medium. We suppose further that the displacement is transferred entirely to the neighboring portions without influence of the curvature. (This situation occurs, for example, if $\mathcal{M}$ is a surface and the medium is distributed on and moving along $\mathcal{M}$ without friction or obstruction.) Then, the displacement D should enjoy the "wave equation" $D_{tt} = \triangle D$ (by taking an appropriate scale), therefore $-\triangle f = \lambda f$ describes the standing wave $D = e^{i\sqrt{\lambda}t} f(x)$.

We notice that the *total energy* $\int_{\mathcal{M}} (|D_t|^2 + |\nabla D|^2) d\mathcal{M}$ is finite if and only if $\int_{\mathcal{M}} |f|^2 d\mathcal{M}$ is finite. Therefore, what we are asking is the conditions for $\mathcal{M}$ not to admit a standing wave of finite energy.

Before describing the general statement, let us see examples of $\mathcal{M}$ which have L^2-solutions.

Received December 7, 1992.