Advanced Studies in Pure Mathematics 21, 1992
Zeta Functions in Geometry
pp. 281–325

On Hermitian Forms attached to Zeta Functions

Hiroyuki Yoshida

§0. Introduction

In this paper, we shall deal with some problems of analysis which arise naturally from explicit formulas. For $F \in C_c^\infty(\mathbf{R})$, set

$$\Phi(s) = \int_{-\infty}^{\infty} F(x) e^{(s-1/2)x}\, dx, \quad s \in \mathbf{C}, \qquad \hat{F}(t) = \Phi(\frac{1}{2} + it), \quad t \in \mathbf{R}.$$

Then the explicit formula for $\zeta(s)$ reads as

$$\begin{aligned}\sum_\rho \Phi(\rho) = &\int_{-\infty}^{\infty} F(x)(e^{x/2} + e^{-x/2})dx - (\log \pi)F(0) \\ &- \sum_p \sum_{m=1}^{\infty} \frac{\log p}{p^{m/2}} (F(m \log p) + F(-m \log p)) \\ &+ \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{F}(t) \operatorname{Re}(\psi(\frac{1}{4} + \frac{it}{2}))\, dt,\end{aligned}$$

where $\psi(s) = \Gamma'(s)/\Gamma(s)$ and ρ extends over all non-trivial zeros of $\zeta(s)$. The functional $T(F) = \sum_\rho \Phi(\rho)$ defines a distribution on $\mathbf{R}$. A well known observation of Weil states that T is positive definite i.e. $T(\alpha * \tilde{\alpha}) \geq 0$ for every $\alpha \in C_c^\infty(\mathbf{R})$ if and only if the Riemann hypothesis holds for $\zeta(s)$. We can define a hermitian form $\langle\ ,\ \rangle$ on $C_c^\infty(\mathbf{R})$ by

$$\langle \varphi_1, \varphi_2 \rangle = T(\varphi_1 * \tilde{\varphi}_2), \qquad \varphi_1, \varphi_2 \in C_c^\infty(\mathbf{R}).$$

For $a > 0$, we set

$$C(a) = \{\ \varphi \in C_c^\infty(\mathbf{R}) \mid \ \operatorname{supp}(\varphi) \subseteq [-a, a]\ \}.$$

Then R.H. is equivalent to the positive definiteness of $\langle\ ,\ \rangle | C(a)$ for every $a > 0$ (cf. Proposition 2). It can easily be verified that $\langle\ ,\ \rangle | C(a)$ is positive definite if a is sufficiently small. Now we can naturally ask:

Received September 3, 1990.