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Zeta Functions in Geometry
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Spectrum and Geodesic Flow

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§0. Introduction

The zeta function of concern in this paper is the spectral zeta function $\zeta_\Delta(s)$ of a compact, riemannian manifold (M, g). It is defined by:

$$\zeta_\Delta(s) = \sum_{j=1}^{\infty} \lambda_j^{-s} \qquad (\mathrm{Re}\, s > \frac{1}{2}n),$$

where $n = \dim M$, and where $\{0 = \lambda_0 < \lambda_1 \leqq \cdots \leqq \uparrow \infty\}$ is the set of eigenvalues of the (positive) Laplacian Δ of (M, g). This set is called the spectrum of (M, g) and is denoted by $\mathrm{Spec}(M, g)$. ζ_Δ has a meromorphic continuation of to all $\mathbb{C}$, with simple poles at the points $s_j = \frac{n-j}{2}$ such that $s_j \neq 0, -1, -2, \ldots.$ The residues $\mathrm{Res}_{s=s_j} \zeta_\Delta$ are perhaps the most classical spectral invariants. They are given by integrals of local geometric invariants of (M, g). More precisely, $\mathrm{Res}_{s=s_j} \zeta_\Delta = \int_M P_j(R, \nabla R, \ldots)\, d\,\mathrm{vol}$, where P_j is a polynomial in the curvature tensor R and its covariant derivatives [Gi]. The question naturally arises: to what extent is (M, g) determined by $\mathrm{Spec}(M, g)$?

It is of course well-known that $\mathrm{Spec}(M, g)$ does not always determine (M, g) up to isometry. Indeed, quite a variety of isospectral pairs is known at present (see [Su]). However, each known pair is quite special: for example, each isospectral pair has a common riemannian cover, and in most (if not all) cases, the manifolds have multiple length spectra. Here, the length spectrum $\mathrm{Lsp}(M, g)$ of (M, g) is the set of lengths L_{γ_j} of its closed geodesics γ_j. So one asks:

Question 1. Does $\mathrm{Spec}(M, g)$ determine (M, g) up to local isometry? Do isospectral pairs have a common riemannian cover?

Question 2. Does $\mathrm{Spec}(M, g)$ determine the generic (M, g) up to isometry? For example, if $\mathrm{Lsp}(M, g)$ is simple, is (M, g) spectrally determined?

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