Advanced Studies in Pure Mathematics 18-II, 1990
Kähler Metrics and Moduli Spaces
pp. 11–83

Einstein-Kähler Metrics with Positive Ricci Curvature

A. Futaki, T. Mabuchi and Y. Sakane

Introduction

The affirmative answer, by Aubin [3] and Yau [77] , to Calabi's conjecture on the existence and uniqueness of Einstein-Kähler metrics in the Ricci flat or negative case has many important consequences not only in differential geometry but also in algebraic geometry (see for instance Bourguignon et al. [17]). Their proofs fail, however, for compact complex connected manifolds M with positive first Chern class $c_1(M) > 0$, and actually a couple of obstructions are known so far concerning the existence of such metrics. The purpose of this paper is to give a brief survey of the recent progress on the existence and uniqueness problem of Einstein-Kähler metrics for M as above.

In 1957, Matsushima [63] obtained the reductiveness of the group $\mathrm{Aut}(N)$ of holomorphic automorphisms of compact connected Einstein-Kähler manifolds N with $c_1(N) > 0$, by showing that the Lie algebra $H^0(N, \mathcal{O}(TN))$ of holomorphic vector fields is a complexification of the Lie algebra of Killing vector fields on N. This, for instance, enabled Yau [76] to construct examples of compact Kähler manifolds with $c_1 > 0$ carrying no Einstein-Kähler metrics. In Section 1, we shall discuss these results together with Kobayashi's semistability of tangent bundle for compact Einstein-Kähler manifolds.

Inspired by the result of Kazdan and Warner [42, 43] on Nirenberg's problem, Futaki [25] found another obstruction $\mathcal{F}\colon H^0(M, \mathcal{O}(TM)) \to \mathbb{C}$ in 1983. This in particular allowed him to construct an example of M with $\mathcal{F} \neq 0$ which admits no Einstein-Kähler metrics but has reductive $\mathrm{Aut}(M)$. A quick review of this Futaki's obstruction $\mathcal{F}$ will be given in Section 2, while we study another aspect of $\mathcal{F}$ in Section 3 from a viewpoint of symplectic geometry, following Futaki [28] and Mabuchi [58]. In

Received November 13, 1988.
Revised April 19, 1989.