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Behavior of the Zeta-Function of Open Surfaces at $s=1$

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Dedicated to Professor Kenkichi Iwasawa

A major theme in the work of Iwasawa is the interplay between theorems and conjectures concerning zeta-functions in the number-field case with analogous theorems and conjectures in the function-field case. A particularly striking example of this was provided by Tate and M. Artin, who considered the function-field analogue of the conjecture of Birch and Swinnerton-Dyer, and largely showed that this conjecture was equivalent to a conjecture about the zeta-function of certain complete non-singular surfaces X over finite fields [T]. They also showed that this conjecture ($Z(X, 1)=\pm\chi(X, G_a)/\chi(X, G_m)$ in the notation of this paper) was true if and only if the Brauer group $H^2(X, G_m)$ of X was finite (which it may always be, as far as we know).

However, the number-theoretic case in someways resembles more closely the case where the surface, although still non-singular, is no longer complete. In this paper, we consider an open subset U of X obtained by removing a curve C, and show that the analogous conjecture ($Z(U, 1)=\pm\chi(X, G_a^U)/\chi(X, G_m^U)$) remains true, if the Brauer group of X is finite.

The reader should be cautioned that, because of a 2-torsion defect in [L2], all theorems are only valid up to 2-torsion groups or powers of 2, as the case may be.

§ 1. Definition and properties of Euler characteristics

We begin with some algebraic preliminaries. Let $\mathscr{P}$ be the abelian category whose objects are given by triples consisting of two finitely-generated abelian groups A, A' of the same rank and a non-degenerate bilinear map $\langle\ ,\ \rangle_A$: $A\times A'\to\boldsymbol{Q}$. A morphism from $(A, A', \langle\ ,\ \rangle_A)$ to $(B, B', \langle\ ,\ \rangle_B)$ is a pair of morphisms α: $A\to B$ and β: $B'\to A'$ such that $\langle\alpha(a), b'\rangle_B=\langle a, \beta(b')\rangle_A$ for all $a\in A$, $b'\in B'$.

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