Advanced Studies in Pure Mathematics 13, 1988
Investigations in Number Theory
pp. 193–214

On an Application of Zagier's Method in the Theory of Selberg's Trace Formula

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Introduction

Let H be the complex upper half plane, and put $G=PSL(2, \boldsymbol{R})$, $\Gamma=PSL(2, \boldsymbol{Z})$. Then, the well-known Selberg trace formula holds for the Hilbert space $L^2(\Gamma\backslash H)$. Let furthermore $\omega: z\to -\bar{z}$ be the reflection with respect to the imaginary axis, and let $\tilde{G}=\langle G, \omega\rangle$ be the group generated by G and ω. Then, the triple $(\tilde{G}, H, 1)$ turns out to be a weakly symmetric Riemannian space in the notation of Selberg (§ 1). Therefore, it is possible to investigate the trace formula for the Hilbert space $L^2(\tilde{\Gamma}\backslash H)$ with $\tilde{\Gamma}=\langle\Gamma, \omega\rangle$.

The space $L^2(\Gamma\backslash H)$ has the direct sum decomposition $L^2(\Gamma\backslash H)=V_e\oplus V_o$, where V_e and V_o are defined by $V_e=\{f\in L^2(\Gamma\backslash H)|f(\omega z)=f(z)\}$, $V_o=\{f\in L^2(\Gamma\backslash H)|f(\omega z)=-f(z)\}$ respectively, in accordance with the operation of ω. Since it is clear that $V_e=L^2(\tilde{\Gamma}\backslash H)$, the trace formulas for $L^2(\tilde{\Gamma}\backslash H)$ and for V_e are the same.

In fact, Venkov [8: Chap. 6] presented trace formulas for V_e and V_o in more general cases where the discontinuous group has an ω-invariant fundamental domain.

On the other hand, Zagier [10] gave a new method to derive the trace formulas in the case of $\Gamma=PSL(2, \boldsymbol{Z})$, considering an integral of the form $I(s)=\int_{\Gamma\backslash H}K_0(z, z)E(z, s)dz$ (§ 2).

In the present paper, we shall prove the trace formula for V_e, i.e., for $L^2(\tilde{\Gamma}\backslash H)$ by means of Zagier's method in the case of $\Gamma=PSL(2, \boldsymbol{Z})$ (§ 3 and Theorem 2), and add an explicit form of the trace formula for V_o as a direct consequence of the trace formulas for $L^2(\Gamma\backslash H)$ and V_e (Theorem 3).

§ 1. Weakly symmetric Riemannian space

Let S be a Riemannian manifold with a positive definite metric $ds^2=\sum g_{ij}dx^i dx^j$. The mapping of S onto itself is called an isometry if it holds

Received February 4, 1986.