Toda Lattice Hierarchy

Kimio Ueno and Kanehisa Takasaki

Introduction

In the last decade the theory of completely integrable non linear systems, the so called "soliton theory", has made remarkable progress, in which intensive researches have been done by many physicists and mathematicians.   Among them the Toda lattice [36] has always been, together with the Korteweg-de Vries (*KdV*) equation, one of the most classical and important objects to be investigated from various points of view, both physical and mathematical.

Several varieties of methods have been developed to reveal the profound mathematical structure in the Toda lattice: Inverse scattering method, spectral theory, Bäcklund transform [5, 7, 9, 18, 27, 37], algebro-geometric method [3, 5, 6, 7, 8, 13, 28, 29, 30], Hirota's method [10, 11, 19], orbit method, group representation theory [2, 3, 4, 14, 15, 16, 17, 30, 31, 32, 35].

In the present paper, inspired by the recent developments in the study on the Kadomtsev-Petviashvili (*KP*) hierarchies [20–25, 34], a hierarchy (a series of mutually commutative higher evolutions) for the two dimensional infinite Toda lattice is introduced.   Its algebraic structure, the linearization, the bilinearization in terms of the τ function, the reductions and the special solutions are investigated in detail.   Also its analogues of the B and C types and the multi-component type are considered.   Our method, which is closely related with those used in [12, 20–26, 33, 34], has the advantage of making the treatment of the infinite lattice extremely clear and algebraic.

Our investigation in the present paper is motivated by the following observations:

The two dimensional infinite Toda lattice (hereafter we shall call it simply the "Toda lattice" (*TL*)) is, by definition, the non linear wave equation

$$(0.1) \qquad \partial_{x_1}\partial_{y_1}u(s) = e^{u(s)-u(s-1)} - e^{u(s+1)-u(s)},$$