Chapter 7

Pyramidal Traveling Fronts

Pyramidal traveling fronts have been studied by [44, 45, 28] and so on for bistable reaction-diffusion equations in $\mathbb{R}^n$. See also Hamel and Nadirashvili [23] for pyramidal traveling fronts in the Fisher–KPP equations in $\mathbb{R}^n$. In this chapter we study traveling fronts of pyramidal shapes to bistable reaction-diffusion equations in $\mathbb{R}^n$ following [44, 45, 28]. Here $n \geq 3$ is a given integer. We study

$$\begin{aligned} u_t &= \Delta u + f(u), \quad \boldsymbol{x} \in \mathbb{R}^n, t > 0, \\ u(\boldsymbol{x}, 0) &= u_0(\boldsymbol{x}), \quad \boldsymbol{x} \in \mathbb{R}^n. \end{aligned} \tag{7.1}$$

Here u_0 is a given bounded and uniformly continuous function. In this chapter we assume

(A1) f is of class C^1 in some open interval including $[-1, 1]$. It satisfies $f(1) = 0$, $f(-1) = 0$, $f'(-1) < 0$, and $f'(1) < 0$;

(A2) f satisfies $\int_{-1}^{1} f(u)\,\mathrm{d}u > 0$;

(A3) there exist $k > 0$ and Φ such that one has

$$\begin{gathered} -\Phi''(x) - k\Phi'(x) - f(\Phi(x)) = 0, \qquad x \in \mathbb{R}, \\ -\Phi'(x) > 0, \qquad x \in \mathbb{R}, \\ \Phi(-\infty) = 1, \quad \Phi(+\infty) = -1. \end{gathered}$$

7.1 Preliminaries for Pyramidal Traveling Fronts

Let

$$\beta = \frac{1}{2} \min\{-f'(-1), -f'(1)\} > 0.$$

There exists a constant $\delta_* \in (0, 1/4)$ with

$$-f'(s) > \beta \quad \text{if} \quad |s + 1| < 2\delta_* \text{ or } |s - 1| < 2\delta_*.$$

There exist constants $K_0 > 0$ and $\kappa_0 > 0$ such that one has

$$\max\{|\Phi'(x)|, |\Phi''(x)|, |x\Phi'(x)|\} \leq K_0 \exp(-\kappa_0|x|) \qquad \text{for all } x \in \mathbb{R}.$$

Let $c \in (k, \infty)$ be arbitrarily given. Without loss of generality, we assume that a traveling front is moving to the x_n-direction. We write $\boldsymbol{x} = (x_1, \ldots, x_n)$ and $\boldsymbol{x}' = (x_1, \ldots, x_{n-1})$. Now we put $s = x_n - ct$ and $u(\boldsymbol{x}, t) = w(\boldsymbol{x}', s, t)$. Denoting $w(\boldsymbol{x}', s, t)$ simply by $w(\boldsymbol{x}, t)$, we have

$$\begin{aligned} w_t - \Delta w - c\mathrm{D}_n w - f(w) = 0, \quad \boldsymbol{x} \in \mathbb{R}^n, t > 0, \\ w(\boldsymbol{x}, 0) = u_0(\boldsymbol{x}), \quad \boldsymbol{x} \in \mathbb{R}^n. \end{aligned} \tag{7.2}$$