Chapter 4

Traveling Fronts for Bistable Nonlinearity II

In this chapter we prove the existence of traveling front solutions to general bistable reaction-diffusion equations. The argument adopted here is different from that of Chapter 1. We study a reaction-diffusion equation with bistable nonlinear term

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + f(u), \qquad x \in \mathbb{R},\ t > 0, \tag{4.1}$$

$$u(x,0) = u_0(x), \qquad x \in \mathbb{R}. \tag{4.2}$$

Here u_0 is a given bounded uniformly continuous function from $\mathbb{R}$ to $\mathbb{R}$. Let $u(x,t;u_0)$ be the solution of (4.1) and (4.2).

We state the standing assumption on the nonlinear term f in this chapter. A function f is of class C^1 in an open interval including $[-1,1]$. Let $F(s)$ satisfy $F'(s) = f(s)$. It satisfies $f(-1) = 0$, $f(1) = 0$, $f'(-1) < 0$, $f'(1) < 0$.

Let

$$\beta = \frac{1}{2}\min\{-f'(-1), -f'(1)\} > 0.$$

Let $\delta_* \in (0,1)$ be small enough so that one has

$$\min_{|s+1|\le 2\delta_*}(-f'(s)) > \beta, \quad \min_{|s-1|\le 2\delta_*}(-f'(s)) > \beta.$$

We put

$$M = \max_{|s|\le 1+2\delta_*}|f'(s)|.$$

If $\Phi(y)$ satisfies $-\Phi''(y) - c\Phi'(y) - f(\Phi(y)) = 0$ for all $y \in \mathbb{R}$ with some $c \in \mathbb{R}$, then Φ is called a traveling wave. Here we study traveling waves connecting -1 and 1, and impose the following condition

$$\Phi(-\infty) = 1, \quad \Phi(\infty) = -1.$$

Thus we study

$$\begin{aligned} -\Phi''(y) - c\Phi'(y) - f(\Phi(y)) = 0, \qquad y \in \mathbb{R}, \\ \Phi(-\infty) = 1, \quad \Phi(\infty) = -1. \end{aligned} \tag{4.3}$$

If one puts $\phi(y) = \Phi(-y)$, ϕ becomes a traveling wave with $\phi(-\infty) = -1$, $\phi(\infty) = 1$ with speed $-c$. Let u_0 be continuously differentiable and satisfy

$$-u_0'(x) > 0, \qquad x \in \mathbb{R}, \tag{4.4}$$

$$u_0(-\infty) = 1, \quad u_0(+\infty) = -1. \tag{4.5}$$