Chapter 2

Existence and the Schauder Estimates of Solutions

In this chapter we show the existence of a solution to reaction-diffusion equation and give the Schauder estimates on the solution. We will use these facts in later chapters.

2.1 Existence and Uniqueness of a Solution

Let $n \geq 1$ and $m \geq 1$ be given integers. We define

$$X = \{\text{bounded and uniformly continuous functions from } \mathbb{R}^n \text{ to } \mathbb{R}^m\} \tag{2.1}$$

with a norm

$$\|v\| = \sup_{x \in \mathbb{R}^n} |v(x)|.$$

Then X is a Banach space as is shown in Lemma 2.3.

For any $d_j > 0$ $(1 \leq j \leq m)$ let

$$u(x,t) = \begin{pmatrix} u_1(x,t) \\ \vdots \\ u_m(x,t) \end{pmatrix}, \quad D = \begin{pmatrix} d_1 & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & d_m \end{pmatrix}.$$

A heat equation

$$u_t = D\Delta u \qquad x \in \mathbb{R}^n,\ t > 0$$

with a given initial function $u(x,0)$ is solved as

$$u(x,t) = \int_{\mathbb{R}^n} K(x-y,t)u(y,0)\,\mathrm{d}y.$$

Here $K(x,t)$ is the heat kernel given by

$$K(x,t) = \begin{pmatrix} K_1(x,t) & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & K_m(x,t) \end{pmatrix},$$

where

$$K_j(x,t) = \frac{1}{(4\pi d_j t)^{\frac{n}{2}}} \exp\left(-\frac{|x|^2}{4d_j t}\right).$$

For any $u_0 \in X$ and any $t \geq 0$, we set

$$\left(T(t)u_0\right)(x) = \int_{\mathbb{R}^n} K(x-y,t)u_0(y)\,\mathrm{d}y.$$

Then $\{T(t)\}_{t \geq 0}$ is called a set of evolution operators and satisfies