Chapter 1

Phase Plane Analysis

In this chapter we prove the existence of traveling fronts by using phase plane analysis. One can see [2, 3, 14, 15, 35] for this analysis. We study a reaction-diffusion equation with a nonlinear term

$$\begin{aligned} \frac{\partial u}{\partial t} &= \frac{\partial^2 u}{\partial x^2} + f(u), \qquad x \in \mathbb{R},\ t > 0, \\ u(x,0) &= u_0(x), \qquad x \in \mathbb{R}. \end{aligned} \tag{1.1}$$

Here u_0 is a given function that is a bounded and uniformly continuous function from $\mathbb{R}$ to $\mathbb{R}$. Let $u(x,t;u_0)$ be the solution of (1.1). The standing assumption on f in this chapter is as follows. A function f is of class $C^1[b,1]$ for $b \in \mathbb{R}$ with $b < 1$ and it satisfies

$$\begin{aligned} f(1) = 0, \quad f'(1) < 0, \\ f(s) > 0 \qquad \text{for} \quad b < s < 1. \end{aligned} \tag{1.2}$$

Additional assumptions on f will be stated in the following sections.

For $c \in \mathbb{R}$ we put $y = x - ct$ and $w(y,t) = u(x,t)$. Then we have

$$\begin{aligned} w_t - cw_y - w_{yy} - f(w) = 0, &\quad y \in \mathbb{R},\ t > 0, \\ w(y,0) = u_0(y), &\quad y \in \mathbb{R}. \end{aligned} \tag{1.3}$$

Let U be the profile of a traveling front. Then U is an equilibrium solution of (1.3) and satisfies

$$-U''(y) - cU'(y) - f(U(y)) = 0, \qquad y \in \mathbb{R}. \tag{1.4}$$

Equation (1.4) is called the profile equation of a one-dimensional traveling front U.

To find (c, U) that satisfies (1.4), we introduce the following equation

$$\begin{aligned} p'(z) &= -c - \frac{f(z)}{p(z)}, \qquad b < z < 1, \\ p(z) &< 0, \qquad b < z < 1, \\ p(1) &= 0 \end{aligned} \tag{1.5}$$

for every $c \in \mathbb{R}$. If $p(z)$ satisfies (1.5), we set

$$y = \int_b^U \frac{\mathrm{d}z}{p(z)},$$

and have

$$\frac{\mathrm{d}y}{\mathrm{d}U} = \frac{1}{p(U)}, \quad U'(y) = p(U(y)) < 0,$$
$$U''(y) = p'(U(y))U'(y) = -cp(U(y)) - f(U(y)) = -cU'(y) - f(U(y)).$$