Chapter 6

Applications of the Parametrisation

6.1 Structures of finite area

We seek now to determine necessary and sufficient conditions for a framed convex projective structure to have finite area, given its representative in $\mathbb{R}_{>0}^{\triangle \cup \underline{E}}$. This leads to a new proof of a result from Marquis [38].

Throughout this section, the surface $S = S_{g,n}$ will be of negative Euler characteristic, with at least one puncture and orientation ν. Fix a positive framed convex projective structure on S with holonomy representation hol and developing map dev. We denote the holonomy group by Γ. Furthermore, fix an ideal triangulation T of S, let $\Omega := \mathrm{dev}(\widetilde{S}) \subset \mathbb{RP}^2$ and denote by $\widetilde{\mathrm{T}}$ the lift of T to Ω. The map $\phi = \phi^+_{\mathrm{T},\nu}$ is the canonical isomorphism defined in Theorem 4.4.1. We identify the simplices $e_{ij}, t_{ijk} \in \mathrm{T}$ with their coordinates $\phi(x)(e_{ij})$ and $\phi(x)(t_{ijk})$ respectively, so as to simplify notation.

Throughout this section, if $\mathcal{C} \subset \mathbb{RP}^2$ is a properly convex domain and $U \subset \mathcal{C}$ then we denote by $\nu_{\mathcal{C}}(U)$ the Hausdorff measure of U with respect to the Hilbert metric on $\mathcal{C}$.

For any properly convex domain $\Omega \subset \mathbb{RP}^2$ and point q, let

$$B_q^{\Omega}(r) = \{x \in \Omega \mid d_{\Omega}(x, q) \leqslant r\}$$

denote the closed metric ball centered at q, of radius r with respect to the Hilbert metric measured on Ω. We will require the following result in order to prove Lemma 6.1.2.

Lemma 6.1.1. *Let $\Omega \subset \mathbb{RP}^2$ be a properly convex domain and let $p \in \Omega$. Then $\nu_{\Omega}(B_p^{\Omega}(r))$ is bounded below by a strictly positive constant depending on r but not on Ω or p.*

Proof. We send Ω to Benzécri position with respect to p, as defined in Theorem 2.3.3. Let $\mathcal{A}$ be the affine patch determined by the construction in Theorem 2.3.3 and just as in that theorem let $B(1)$ and $B(14)$ be the closed balls centred at the origin of radius 1 and 14 respectively with respect to a chosen Euclidean metric on $\mathcal{A}$. In particular, we have

$$B(1) \subset \Omega \subset B(14).$$