Chapter 5

Properties of the Parametrisation

5.1 Change of coordinates

Let S be a surface as in Theorem 4.4.1. Having fixed a triangulation T and orientation ν, the space $\mathcal{T}_3^+(S)$ can be canonically parametrised by positive real numbers. A different choice of ν or T may be interpreted as a change of coordinates. Similarly, the duality map $\sigma : \mathcal{T}_3^+(S) \to \mathcal{T}_3^-(S)$, defined by taking a structure to its dual, induces an involution on Fock-Goncharov moduli space. We explicitly construct these transition maps.

We remark that also the action of $\mathrm{Sym}(3)^n$ on $\mathcal{T}_3^+(S)$ descends to a change of coordinates. However, even in the simplest cases, the transition functions are quite convoluted and we do not have a local description of them.

5.1.1 Transition maps for a different orientation

The transition map associated to a switch in the orientation of S is simple to describe. Denote by $-\nu$ the opposite orientation of ν. Then for all $q \in \triangle \cup \underline{E}$,

$$\phi_{\mathrm{T},-\nu}(x)(q) = \frac{1}{\phi_{\mathrm{T},\nu}(x)(q)}.$$

Indeed triple ratios are computed with respect to flags with the opposite cyclical order, and edge ratios are computed after permuting the second and final arguments.

5.1.2 Duality map in Fock-Goncharov coordinates

Fix a triangulation T and an orientation of S. Two dual structures $x \in \mathcal{T}_3^+(S)$ and $y \in \mathcal{T}_3^-(S)$, are generally represented by different points of $\mathbb{R}_{>0}^{\triangle\cup\underline{E}}$. That is, the composition map

$$\phi_{\mathrm{T}}^- \circ \sigma^+ \circ (\phi_{\mathrm{T}}^+)^{-1} = \phi_{\mathrm{T}}^+ \circ \sigma^- \circ (\phi_{\mathrm{T}}^-)^{-1} \colon \mathbb{R}_{>0}^{\triangle\cup\underline{E}} \to \mathbb{R}_{>0}^{\triangle\cup\underline{E}}$$

is not trivial. Recall from §4.6 that duality maps a flag (V, η) to its dual $(\eta^\perp, V^\perp)$. Referring to the left hand side of Figure 5.1, a straightforward calculation shows that the change of coordinates is locally:

$$e_{20} \mapsto \frac{e_{02}t_{012}(t_{023}+1)}{t_{012}+1}, \qquad e_{02} \mapsto \frac{e_{20}t_{023}(t_{012}+1)}{t_{023}+1} \qquad \text{and} \qquad t_{012} \mapsto \frac{1}{t_{012}}.$$