Chapter 4

The Parametrisation of the Moduli Space

Ratios of flags can be viewed as the algebraic underpinning of Fock and Goncharov's parametrisation of the moduli spaces of interest in these notes. In addition, the parametrisation is based in geometry and topology through framings of the ends and ideal triangulations. We will now describe these ingredients and then give a self-contained construction of the Fock-Goncharov parametrisation. Our proofs in this section are different from those found in [15].

4.1 Framing the ends

We described the spaces $\mathcal{T}_3^+(S)$ and $\mathcal{T}_3^-(S)$ corresponding to framed structures where each end is either *minimal* or *maximal*, and the distinction between the spaces arises from the framings that are allowed for a hyperbolic end. In this section, we give a different definition of these spaces and show in the proofs of Theorems 4.4.1 and 4.7.1 that these definitions are in fact equivalent. We refer the reader to the classification of ends as hyperbolic, special and cusps, as well as the definition of maximal and minimal hyperbolic ends given in §2.2.8.

Each end may be endowed with a *framing*. There are two definitions which are related by projective duality. A *positive framing* of $(\Omega, \Gamma, f) \in \mathcal{T}_3^m(S)$ is a choice, for each end $\mathcal{E}_k$, of a pair consisting of:

- If $\mathcal{E}_k$ is a maximal hyperbolic end: the Γ–orbit of the saddle point V of Γ_k and the Γ–orbit of a Γ_k–invariant supporting line η through V.
- Otherwise: the Γ–orbit of a fixed point V of Γ_k in the frontier of Ω, and the Γ–orbit of a Γ_k–invariant supporting line η through V.

Under this definition, there are two possible positive framings for each maximal hyperbolic end, for each minimal hyperbolic end there are four, and for each special end there are three. Each cusp has a unique positive framing. We recall that $\mathcal{T}_3^\times(S)$ is the space of framed structures on S. Hence two positively framed structures are equivalent if and only if they represent the same element of $\mathcal{T}_3^\times(S)$.

We introduce the following notation. We denote by $\mathcal{T}_3^+(S)$ the set of equivalence classes of *positively framed marked properly convex projective structures with maximal or minimal hyperbolic ends* on S.

We call π^+ the natural projection of $\mathcal{T}_3^+(S)$ onto $\mathcal{T}_3^m(S)$ defined by forgetting the framing. A structure (Ω, Γ, f) with a fixed positive framing will be denoted by $(\Omega, \Gamma, f)^+$.