Chapter 3

Ratios and Configurations of Flags

This section is devoted to the development of some foundational tools: triple ratios, cross ratios, quadruple ratios and configurations of flags. Most of the definitions and conventions follow Fock and Goncharov [15].

3.1 Flags

Recall from §2.1.7 that the analytic homomorphism $\mathrm{GL}(3,\mathbb{R}) \to \mathrm{SL}(3,\mathbb{R})$ defined by

$$A \mapsto (\det(A))^{-1/3} A$$

descends to an isomorphism $\mathrm{PGL}(3,\mathbb{R}) \to \mathrm{SL}(3,\mathbb{R})$. We work with $\mathrm{SL}(3,\mathbb{R})$, which allows us to talk about eigenvalues of maps, and it will be clear from context whether an element of $\mathrm{SL}(3,\mathbb{R})$ acts on $\mathbb{RP}^2$ or on $\mathbb{R}^3$. We also remark that the action on $\mathbb{R}^3$ is by orientation preserving maps.

A *flag* $\mathcal{F}_i := (V_i, \eta_i)$ of $\mathbb{RP}^2$ is a pair consisting of a point $V \in \mathbb{RP}^2$ and a line $\eta \subset \mathbb{RP}^2$ passing through V. An m–tuple of flags $\{\mathcal{F}_1, \ldots, \mathcal{F}_m\}$ is in *general position* if all of the following conditions are satisfied:

- no three points are collinear;
- no three lines are concurrent;
- $\eta_i(V_j) = 0 \iff i = j$.

Henceforth, we will denote by $(\!(\mathcal{F}_1, \ldots, \mathcal{F}_m)\!)$ a cyclically ordered m–tuple of flags, as opposed to an ordered m–tuple $(\mathcal{F}_1, \ldots, \mathcal{F}_m)$. The group of projective transformations acts on the space of flags via:

$$T \cdot \mathcal{F}_i := (T \cdot V_i, T \cdot \eta_i), \qquad \text{for all } T \in \mathrm{SL}(3,\mathbb{R}).$$

The action naturally extends to m–tuples, ordered m–tuples and cyclically ordered m–tuples of flags. Furthermore, the action preserves the respective subspaces of flags in general position.

Lemma 3.1.1. *The action of* $\mathrm{SL}(3,\mathbb{R})$ *on the space of ordered m–tuples of flags in general position is free if and only if* $m \geqslant 3$.

Proof. The cases $m = 1, 2$ are straight forward.

Suppose $m \geqslant 3$ and let $\mathfrak{F} = (\mathcal{F}_1, \ldots, \mathcal{F}_m)$ be an ordered m–tuple of flags in general position, where $\mathcal{F}_i := (V_i, \eta_i)$. Then the ordered 4–tuple of points $(V_1, V_2, V_3, \eta_1 \cap \eta_2)$ is in general position and its stabiliser in $\mathrm{SL}(3,\mathbb{R})$ is trivial.