Chapter 2
Projective Structures on Surfaces

According to Felix Klein's influential Erlanger program of 1872,

Geometry is the study of the properties of a space,
which are invariant under a group of transformations.

In the framework of Klein, the familiar Euclidean geometry of the plane consist of the 2–dimensional Euclidean space and its group of isometries. In general, a *geometry* is a pair (G, X), where X is a (sufficiently nice) space and G is a (sufficiently nice) group acting on the space. Geometric properties are precisely those that are preserved by the group. A geometry in Klein's sense may not allow the concepts of distance or angle that are the familiar starting points of Euclidean geometry. Examples of such geometries include affine geometry and projective geometry.

If $H \leqslant G$ is a subgroup, then H corresponds to the group of transformations of a more *rigid* structure on X. This is called a *stiffening* of (G, X), since an (H, X) structure inherits the geometric properties of (G, X), but may have more. Examples of stiffenings are:

$$\text{Euclidean geometry} \xrightarrow{\text{is a stiffening of}} \text{affine geometry} \xrightarrow{\text{is a stiffening of}} \text{projective geometry.}$$

The contents of projective geometry arises from the structure of the underlying vector space. This allows the concepts such as lines, conics and quadrics. Affine geometry allows the additional notions of parallel lines and ratios of distances along parallel lines, and Euclidean geometry allows the additional notions of distance and angle. Figure 2.1 shows a manifestation of stiffening in this context.

A subtle point is that Euclidean and affine transformations leave a subspace of projective space invariant, and this invariant subspace plays a role in the definition of the additional geometric properties. In a similar way, hyperbolic geometry and more general Hilbert geometries are encountered as stiffenings of projective geometry in these notes.

2.1 Projective geometry

The *real projective plane* $\mathbb{RP}^2$ is the space of all lines through the origin in $\mathbb{R}^3$. Scalar multiplication gives an action of the multiplicative group $\mathbb{R}^\times$ on $\mathbb{R}^3\backslash\{0\}$, and we may view $\mathbb{RP}^2$ as the quotient space of this action. A *point* in $\mathbb{RP}^2$ is a line