Chapter 11
Restricted Usage of Vector Fields

In this chapter, we discuss two typical situations where we cannot use all of the vector fields in Γ.

11.1 Systems of nonlinear wave equations with multiple propagation speeds

In this section, we consider systems of wave equations with multiple propagation speeds. Let us consider

$$\Box_{c_j} u_j(t,x) = F_j(u,\partial u), \qquad (t,x)\in(0,\infty)\times\mathbb{R}^3, \tag{11.1}$$

$$u_j(0,x)=\varepsilon f_j(x),\ (\partial_t u)(0,x)=\varepsilon g_j(x), \quad x\in\mathbb{R}^3 \tag{11.2}$$

for $j=1,2,\dots,N$, where $\Box_c=\partial_t^2-c^2\Delta$ for $c>0$, and $c_j>0$ for $j=1,2,\dots,N$. As before, $u=(u_j)$, $\partial u=(\partial_a u_j)$, $f=(f_j)$, and $g=(g_j)$ with $1\le j\le N$ and $0\le a\le 3$.

We can make use of the vector fields S, $\Omega=(\Omega_{jk})$, and $\partial=(\partial_a)$, because we have $[\Box_c,S]=2\Box_c$ and $[\Box_c,\Omega_{jk}]=[\Box_c,\partial_a]=0$ for any $c>0$. However, the Lorentz boost $L=(L_k)$ cannot be used because the commuting relation $[\Box_c,L_k]=2(1-c^2)\partial_t\partial_k$ has no good property when $c\neq 1$. Therefore we have to exploit a vector field method without the Lorentz boost L.

We put $\Gamma_*=(\Gamma_{*,j})_{0\le j\le (n^2+n+2)/2}=(S,\Omega,\partial)$, and

$$|\phi(t,x)|_{*,s}:=\left(\sum_{|\alpha|\le s}|\Gamma_*^\alpha\phi(t,x)|^2\right)^{1/2},\ \|\phi(t,\cdot)\|_{*,s}=\big\||\phi(t,\cdot)|_{*,s}\big\|_{L^2(\mathbb{R}^3)}$$

for a smooth function ϕ and a non-negative integer s. By (5.8) and (5.9), we find that (5.11) and (5.12) stay valid if we replace Γ by Γ_*; hence, for $s\in\mathbb{N}_0$, we obtain

$$C^{-1}|\partial\phi(t,x)|_{*,s}\le\sum_{|\alpha|\le s}|\partial(\Gamma_*^\alpha\phi)(t,x)|\le C|\partial\phi(t,x)|_{*,s}. \tag{11.3}$$

We also find that (5.10) remains true if we replace Γ with Γ_*, and that a similar formula to (5.5) holds.

The following null condition for the multiple speed case in three space dimensions was introduced by Yokoyama [178] (it was also partly suggested in Hanouzet-Joly [32]):