Chapter 10

Weaker Conditions for Global Existence

In this chapter, we study a sufficient condition, which is weaker than the null condition, for global existence of small solutions. First we introduce the notion of the weak null condition due to Lindblad–Rodnianski [127]. Next, restricting our consideration to a simple case of $\Box u = F(\partial u)$ in two and three space dimensions, we establish a global existence theorem under a weaker condition than the null condition. Our condition is closely related to the weak null condition. We also investigate the asymptotic behavior of global solutions under some additional conditions, and we shall see that some global solutions under our weaker sufficient condition are not asymptotically free.

10.1 The weak null condition

Let us consider the following systems of quasilinear wave equations:

$$\Box u = F(u, \partial u, \partial^2 u), \quad (t, x) \in (0, \infty) \times \mathbb{R}^n \tag{10.1}$$

with $n = 2$ or 3, where $u = (u_1, \ldots, u_N)^{\mathrm{T}}$ is an $\mathbb{R}^N$-valued unknown function. In this chapter, we regard $\mathbb{R}^N$-vectors as column vectors.

In the previous chapter, we have proved global existence for small initial data under the null condition in two and three space dimensions. As we have mentioned at the end of the previous chapter, the null condition is also a necessary condition for small data global existence if we consider a single equation (the case where $N = 1$) with nonlinearity F of the form $F = \sum_{a,b} \gamma^{ab}(\partial u)\partial_a \partial_b u$ in three space dimensions; however, it is known that the null condition is not necessary for small data global existence if we consider more general situations. For example, if we allow the nonlinearity to depend also on u, we have global existence of small solutions for the following quasilinear wave equation:

$$\Box u = \gamma(u)\Delta u, \quad (t, x) \in (0, \infty) \times \mathbb{R}^3$$

with a smooth function $\gamma = \gamma(\lambda)$ satisfying $\gamma(0) = 0$, for which the null condition is not satisfied. This result was first established for radially symmetric data by Lindblad [124], and then for general C_0^∞-data by Alinhac [8]. Lindblad [126] generalized this global existence result to

$$\Box u = \sum_{a,b=0}^{3} \gamma^{ab}(u)\partial_a \partial_b u, \quad (t, x) \in (0, \infty) \times \mathbb{R}^3 \tag{10.2}$$

with smooth functions $\gamma^{ab} = \gamma^{ab}(\lambda)$ satisfying $\gamma^{ab}(0) = 0$.