Chapter 9

The Null Condition

In this chapter, we introduce a famous sufficient condition for global existence of solutions for small data, which is called the null condition. The global existence theorems under the null condition will be given for two and three space dimensional cases.

9.1 Estimates for the null forms

We consider the Cauchy problem for the following systems with $n = 2$ or 3:

$$\Box u = F(u, \partial u), \qquad (t, x) \in (0, \infty) \times \mathbb{R}^n, \tag{9.1}$$

$$u(0, x) = \varepsilon f(x),\ (\partial_t u)(0, x) = \varepsilon g(x), \quad x \in \mathbb{R}^n, \tag{9.2}$$

where u is an $\mathbb{R}^N$-valued unknown function, and $F = F(\lambda, \lambda')$ is an $\mathbb{R}^N$-valued C^∞-function on $\mathbb{R}^N \times \mathbb{R}^{(1+n)N}$ satisfying

$$F(\lambda, \lambda') = O(|\lambda|^2 + |\lambda'|^2)$$

around $(\lambda, \lambda') = (0, 0)$. We also suppose that $f,\ g \in C_0^\infty(\mathbb{R}^n; \mathbb{R}^N)$, and that ε is a small positive parameter.

Let $F^{(p)}(\lambda, \lambda')$ be given by (5.42) with $z = (\lambda, \lambda')$. Let us put

$$F^{*,\mathrm{red}}(\omega, X, Y) := \begin{cases} F^{(2),\mathrm{red}}(\omega, X, Y) & \text{when } n = 3, \\ F^{(2),\mathrm{red}}(\omega, X, Y) + F^{(3),\mathrm{red}}(\omega, X, Y) & \text{when } n = 2. \end{cases} \tag{9.3}$$

The estimates in Theorem 8.10 suggests that if $F^{*,\mathrm{red}}(\omega, X, Y) \equiv 0$ for all $\omega \in \mathbb{S}^{n-1}$ and $X, Y \in \mathbb{R}^N$, then the solution exists in much longer time than we expect for the general case. At least this is true for the reduced system: If $F^{*,\mathrm{red}}(\omega, X, Y) \equiv 0$, then the reduced system (8.21) is

$$\partial_t \partial_\sigma v(t, \sigma, \omega) = 0,$$

which apparently has a global solution v in $[1, \infty) \times \mathbb{R} \times \mathbb{S}^{n-1}$. This observation motivates the definition of the *null condition*.

Definition 9.1 (The null condition)**.** Let $n = 2, 3$, and $F^{*,\mathrm{red}}$ be given by (9.3). We say that the nonlinear term $F = F(u, \partial u)$ satisfies the *null condition* if we have

$$F^{*,\mathrm{red}}(\omega, X, Y) = 0, \quad \omega \in \mathbb{S}^{n-1},\ X, Y \in \mathbb{R}^N.$$