Chapter 8

Systems of Reduced Equations

In this chapter, we explain how the vector fields are used in connection with the system of reduced equations due to Hörmander [36, 41]. As an immediate application, we will obtain a detailed version of the lifespan estimate in Theorem 6.7. Further applications to global existence and the asymptotic behavior will be given later in Chapter 10.

8.1 Basic estimates

We are going to figure out the main parts of derivatives of solutions to wave equations by using the vector fields in Γ.

As before, we use the polar coordinates $r = |x|$ and $\omega = (\omega_1, \ldots, \omega_n) = |x|^{-1}x$ for $x \in \mathbb{R}^n$, so that we have $x = r\omega$ and $\partial_r = \sum_{j=1}^n (|x|^{-1}x_j)\partial_j$. We always suppose that $\omega_0 = -1$. We define

$$\partial_\pm := \partial_t \pm \partial_r. \tag{8.1}$$

It follows from (5.18) and (5.19) that

$$\partial_+ = \frac{1}{t+r}(S + L_r). \tag{8.2}$$

This simple identity shows that $\partial_+\phi$ has a special feature of rapid decay if we have a good control of generalized derivatives $\Gamma\phi$:

Lemma 8.1. *There is a universal constant C such that we have*

$$|\partial_+\phi(t,x)| \le C\langle t+r\rangle^{-1}|\Gamma\phi(t,x)| \tag{8.3}$$

for a smooth function $\phi = \phi(t,x)$.

Proof. (8.3) follows immediately from (8.2) when $t+r \ge 1$, while it is a consequence of a trivial estimate $|\partial_+\phi(t,x)| \le 2|\partial\phi(t,x)|$ when $t + r \le 1$. $\square$

Using this inequality, we get the following:

Lemma 8.2. *For $a = 0, 1, \ldots, n$, we have*

$$\partial_a\phi(t,x) = \omega_a(\partial_r\phi)(t,x) + O\big(\langle t+r\rangle^{-1}|\Gamma\phi(t,x)|\big), \tag{8.4}$$

$$r^{(n-1)/2}\partial_a\phi(t,x) = \omega_a\partial_r\big(r^{(n-1)/2}\phi(t,x)\big) + O\big(r^{(n-3)/2}|\phi(t,x)|_1\big) \tag{8.5}$$

for $(t,x) \in [0,T) \times (\mathbb{R}^n \setminus \{0\})$ and $\phi \in C^\infty\big([0,T) \times \mathbb{R}^n\big)$ with $T \in (0,\infty]$.