Chapter 7

Wave Equations with More General Nonlinearity

In this chapter, we consider semilinear wave equations of the form $\Box u = F(u, \partial u)$ in three space dimensions. In the previous chapter, the Klainerman–Sobolev inequality combined with the energy estimate gave us good decay estimates for ∂u; however, since the energy inequality does not provide a natural estimate for $\|u(t)\|_{L^2}$, the Klainerman–Sobolev inequality is not always useful to obtain decay estimates for the solution u.

To start with, we briefly discuss the case of $\Box u = F(u)$. The global existence result due to John [52] will be presented. The proof requires only a weighted L^∞–L^∞ estimate, and the energy estimate is not used. Secondly we consider the case of $\Box u = F(u, \partial u)$. Following Lindblad [123], we obtain the lower bound of the lifespan for the solution to semilinear wave equations with quadratic nonlinearity in three space dimensions.

7.1 Decay estimates of the solution in three space dimensions

For the later purpose, we consider wave equations with general propagation speeds. For $c > 0$, we define $\Box_c := \partial_t^2 - c^2\Delta$. Let us consider the linear wave equation

$$\Box_c u(t,x) = \Phi(t,x), \qquad (t,x) \in (0,T) \times \mathbb{R}^3, \tag{7.1}$$

$$u(0,x) = \varphi(x),\ (\partial_t u)(0,x) = \psi(x), \quad x \in \mathbb{R}^3, \tag{7.2}$$

where $\varphi \in C^3(\mathbb{R}^3)$, $\psi \in C^2(\mathbb{R}^3)$ and $\partial_x^\alpha \Phi \in C\big([0,T) \times \mathbb{R}^3\big)$ for $|\alpha| \le 2$.

For $\nu \in \mathbb{R}$ and $t, r \ge 0$, we put

$$\mathcal{W}_\nu(t,r) := \begin{cases} \langle t+r\rangle^\nu, & \nu < 0, \\ \left(\log\left(1 + \dfrac{\langle t+r\rangle}{\langle t-r\rangle}\right)\right)^{-1}, & \nu = 0, \\ \langle t-r\rangle^\nu, & \nu > 0. \end{cases} \tag{7.3}$$

Lemma 7.1. *For any $\kappa > 0$, there is a positive constant C_κ such that*

$$\frac{1}{r}\int_{|t-r|}^{t+r} \langle\rho\rangle^{-\kappa} d\rho \le C_\kappa \langle t+r\rangle^{-1} \mathcal{W}_{\kappa-1}(t,r)^{-1}, \quad t \ge 0,\ r > 0.$$