Chapter 6

Global Existence and Blow-Up for Small Data

In this chapter, we consider the Cauchy problem for systems of semilinear wave equations of the type

$$\Box u = F(\partial u), \quad (t,x) \in (0,\infty) \times \mathbb{R}^n.$$

Combining the Klainerman–Sobolev inequality with the energy estimates, the global existence of solutions for small data, as well as the estimates for the lifespan of solutions will be derived. We shall also give some examples of nonlinearity for which the solutions blow up in finite time.

6.1 Global existence results

In this section we consider the following system of semilinear wave equations:

$$\Box u = F(\partial u), \qquad (t,x) \in (0,\infty) \times \mathbb{R}^n, \tag{6.1}$$

$$u(0,x) = \varepsilon f(x),\ (\partial_t u)(0,x) = \varepsilon g(x), \quad x \in \mathbb{R}^n, \tag{6.2}$$

where $F = F(\lambda') \in C^\infty(\mathbb{R}^{N(n+1)}; \mathbb{R}^N)$ with

$$F(\lambda') = O(|\lambda'|^p)$$

around $\lambda' = 0$ for some integer p with $p \geq 2$. ε is a small and positive parameter. Our aim in this section is to prove the following.

Theorem 6.1 (Klainerman [90]). *Let $n \geq 2$. Suppose that p satisfies*

$$\frac{(p-1)(n-1)}{2} > 1. \tag{6.3}$$

Then, for any $f, g \in C_0^\infty(\mathbb{R}^n; \mathbb{R}^N)$, there is a positive constant ε_0 such that the Cauchy problem (6.1)–(6.2) *admits a unique global solution*

$$u \in C^\infty\big([0,\infty) \times \mathbb{R}^n; \mathbb{R}^N\big),$$

provided that $0 < \varepsilon \leq \varepsilon_0$.

Remark 6.2. The condition (6.3) is equivalent to

$$p \geq \begin{cases} 2, & \text{when } n \geq 4, \\ 3, & \text{when } n = 3, \\ 4, & \text{when } n = 2. \end{cases}$$