Chapter 5

Vector Fields Associated with Wave Equations

5.1 Introduction

In this chapter, we introduce the vector fields associated with the wave equation, and prove some estimates. Following Klainerman [90], we introduce vector fields

$$\begin{aligned} S &:= t\partial_t + \sum_{j=1}^{n} x_j \partial_j, \\ L_k &:= t\partial_k + x_k \partial_t, \qquad 1 \leq k \leq n, \\ \Omega_{jk} &:= x_j \partial_k - x_k \partial_j, \qquad 1 \leq j, k \leq n. \end{aligned}$$

We put $L = (L_k)_{1 \leq k \leq n}$ and $\Omega = (\Omega_{jk})_{1 \leq j < k \leq n}$. L and Ω are the generators of the Lorentz transforms, while S is sometimes referred to as the *scaling operator*. The Lorentz transform is a space-time transformation which does not change the form of the d'Alembertian. The vector fields L_k's are sometimes referred to as the *Lorentz boosts*. We define

$$\Gamma = (\Gamma_a)_{0 \leq a \leq n_0} = (S, L, \Omega, \partial) = (S, (L_k)_{1 \leq k \leq n}, (\Omega_{jk})_{1 \leq j < k \leq n}, (\partial_a)_{0 \leq a \leq n}),$$

where $n_0 = (n^2 + 3n + 2)/2$. We use a multi-index $\alpha = (\alpha_0, \alpha_1, \ldots, \alpha_{n_0}) \in \mathbb{N}_0^{1+n_0}$ to write $\Gamma^\alpha = \Gamma_0^{\alpha_0} \Gamma_1^{\alpha_1} \cdots \Gamma_{n_0}^{\alpha_{n_0}}$. For a non-negative integer s and a sufficiently smooth function $\phi = \phi(t, x)$, we put

$$|\phi(t,x)|_s := \left(\sum_{|\alpha| \leq s} |\Gamma^\alpha \phi(t,x)|^2 \right)^{1/2}, \tag{5.1}$$

$$\|\phi(t,\cdot)\|_{s,p} := \big\| |\phi(t,\cdot)|_s \big\|_{L^p(\mathbb{R}^n)}, \qquad 1 \leq p \leq \infty. \tag{5.2}$$

We write $\|\phi(t,\cdot)\|_s$ for $\|\phi(t,\cdot)\|_{s,2}$. The family Γ of the vector fields and $\|\cdot\|_{s,p}$, sometimes called the *invariant norm*, were found quite useful in the study of nonlinear wave equations, and the method using these vector fields and invariant norms are called the *vector field method* or the *invariant norm method*. The main ingredients of these vector fields are powerful decay estimates such as the Klainerman–Sobolev inequality and the L^1–L^∞ estimate for the d'Alembertian, due to Klainerman [92] and Hörmander [37].

Throughout this chapter, we suppose that $T \in (0, \infty]$, unless otherwise stated.