# Chapter 4

# Local Existence for Nonlinear Wave Equations

In this chapter, we shall obtain a local existence theorem for semilinear wave equations in the Sobolev spaces. We also show the uniqueness and the finite speed propagation of $C^2$-solutions for systems of nonlinear wave equations.

## 4.1 Preliminary estimates

Suppose that $F = F(z) = \big(F_1(z), \ldots, F_N(z)\big) \in C^\infty(\mathbb{R}^K; \mathbb{R}^N)$ with $F(0) = 0$.

For $s \in \mathbb{N}_0$, we put

$$A_s(\tau) := \sup_{|z| \le \tau} \sum_{|\beta| \le s} |\partial_z^\beta F(z)|, \quad \tau \ge 0.$$

Since we have

$$F_j(z) = \left(\int_0^1 (\nabla_z F_j)(\theta z) d\theta\right) \cdot z, \quad 1 \le j \le N,$$

we get

$$\|F(V)\|_{L^2(\mathbb{R}^n)} \le A_1\big(\|V\|_{L^\infty(\mathbb{R}^n)}\big)\|V\|_{L^2(\mathbb{R}^n)}$$

for any $V = (V_1, \ldots, V_K) \in L^\infty(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$.

Let $V \in \mathscr{B}^{[s/2]}(\mathbb{R}^n) \cap H^s(\mathbb{R}^n)$ with $s \in \mathbb{N}_0$. For $|\alpha| \ge 1$, the Leibniz formula yields

$$\partial_x^\alpha \left(F(V)\right) = \sum_{1 \le m \le |\alpha|} \sum_{|\beta| = m} \sum_{j_1, \ldots, j_m = 1}^{K} \sum_{|\gamma_1| + \cdots + |\gamma_m| = |\alpha|} C_{\boldsymbol{\gamma}, \boldsymbol{j}}^{\alpha, \beta} (\partial_z^\beta F)(V) \prod_{k=1}^m \partial_x^{\gamma_k} V_{j_k},$$

where $\boldsymbol{j} = (j_1, \ldots, j_m)$, $\boldsymbol{\gamma} = (\gamma_1, \ldots, \gamma_m)$, and $C_{\boldsymbol{\gamma}, \boldsymbol{j}}^{\alpha, \beta}$'s are appropriate constants. For $1 \le |\beta| \le s$, we get

$$\big\|(\partial_z^\beta F)(V)\big\|_{L^\infty(\mathbb{R}^n)} \le A_s\big(\|V\|_{L^\infty(\mathbb{R}^n)}\big).$$

For $|\gamma_1| + \cdots + |\gamma_m| = s (\ge 1)$ and $j_1, \ldots, j_m \in \{1, \ldots, K\}$, we have

$$\left\| \prod_{k=1}^m \partial_x^{\gamma_k} V_{j_k} \right\|_{L^2(\mathbb{R}^n)} \le \|V\|_{\mathscr{B}^{[s/2]}(\mathbb{R}^n)}^{m-1} \|V\|_{H^s(\mathbb{R}^n)}$$

for $V \in \mathscr{B}^{[s/2]}(\mathbb{R}^n) \cap H^s(\mathbb{R}^n)$, because only one number among $|\gamma_1|, \ldots, |\gamma_m|$ can exceed $s/2$. If $s \ge n+1$, then, since we have $[s/2] + [n/2] + 1 \le s$, the Sobolev