Chapter 3

Linear Wave Equations

Let $T \in (0, \infty]$. In this chapter, we consider the (forward) Cauchy problem for linear wave equations in n space dimensions:

$$\Box u(t,x) = \Phi(t,x), \qquad (t,x) \in (0,T) \times \mathbb{R}^n, \tag{3.1}$$
$$u(0,x) = \varphi(x),\ (\partial_t u)(0,x) = \psi(x), \quad x \in \mathbb{R}^n. \tag{3.2}$$

Equation (3.1) is called *homogeneous* if $\Phi \equiv 0$, and otherwise *inhomogeneous*. We sometimes refer to the homogeneous wave equation as the *free wave equation*, and its solution as the *free solution*.

We assume that all the functions in the above are real-valued. The case of complex-valued functions can be easily treated by splitting the equations into real and imaginary parts. We concentrate on the forward Cauchy problem, because the backward Cauchy problem for the wave equation on $(-T, 0) \times \mathbb{R}^n$ with the initial data (3.2) can be easily treated through a change of variable $t \mapsto -t$.

First we will obtain the explicit formula for the solution through the Fourier transform. Next we introduce the energy estimates which are quite useful in the study of both linear and nonlinear wave equations. Finally, basic decay estimates of solutions to homogeneous wave equations with compactly supported smooth data will be derived, and the asymptotic behavior of the solution will be discussed.

3.1 The explicit formula for solutions of homogeneous wave equations

3.1.1 One space dimensional case

We start with the easiest case.

Lemma 3.1. *Consider the equation*

$$\begin{cases} (\partial_t + c\partial_x) v(t,x) = \Psi(t,x), & (t,x) \in (0,T) \times \mathbb{R}, \\ v(0,x) = v_0(x), & x \in \mathbb{R}, \end{cases} \tag{3.3}$$

where $c \in \mathbb{R}$, $v_0 \in C^1(\mathbb{R})$, and $\Psi \in C\big([0,T) \times \mathbb{R}\big)$. Then (3.3) admits a unique solution $v \in C^1\big([0,T) \times \mathbb{R}\big)$, and we have

$$v(t,x) = v_0(x - ct) + \int_0^t \Psi\big(\tau, x - c(t-\tau)\big) d\tau, \quad (t,x) \in [0,T) \times \mathbb{R}. \tag{3.4}$$