Chapter 1

Introduction

We consider the Cauchy problem (or the initial value problem) for a system of wave equations

$$(\partial_t^2 - \Delta)u = F(u, \partial u, \partial^2 u), \qquad (t,x) \in (0,T) \times \mathbb{R}^n, \tag{1.1}$$

$$u(0,x) = \varphi(x),\ \partial_t u(0,x) = \psi(x), \quad x \in \mathbb{R}^n, \tag{1.2}$$

where $u = (u_1, \ldots, u_N)$ is an $\mathbb{R}^N$-valued unknown function of $(t,x) \in \mathbb{R} \times \mathbb{R}^n$, and $\Delta = \sum_{k=1}^n \partial_k^2$ is the Laplacian, while ∂u and $\partial^2 u$ denote the first and second derivatives of u. Here we have used the notation $\partial_0 = \partial_t = \partial/\partial t$ and $\partial_k = \partial_{x_k} = \partial/\partial x_k$ for $1 \le k \le n$. More specifically, we write $\partial u = (\partial_a u_j)$ and $\partial^2 u = (\partial_a \partial_b u_j)$ with suffixes $1 \le j \le N$ and $0 \le a, b \le n$.

We assume that the nonlinear term $F = F(\lambda, \lambda', \lambda'')$ is a sufficiently smooth function of $(\lambda, \lambda', \lambda'') \in \mathbb{R}^N \times \mathbb{R}^{(n+1)N} \times \mathbb{R}^{(n+1)^2 N}$, and that $F(0,0,0) = 0$. Here $\lambda = (\lambda_j)$, $\lambda' = (\lambda'_{j,a})$, and $\lambda'' = (\lambda''_{j,ab})$ are variables for which $u = (u_j)$, $\partial u = (\partial_a u_j)$, and $\partial^2 u = (\partial_a \partial_b u_j)$ are substituted, respectively, where the suffixes run as before. To simplify the description, we put $\Box = \partial_t^2 - \Delta$, which is called the *d'Alembertian*.

We say that (1.1) is *semilinear* [1], if the nonlinear term F depends only on $(u, \partial u)$ and is independent of $\partial^2 u$. The system (1.1) is said to be *quasilinear*, if the nonlinear term F is linear in $\partial^2 u$. In other words, (1.1) is quasilinear, if $F = (F_1, \ldots, F_N)$ has the following form:

$$F_j(u, \partial u, \partial^2 u) = \sum_{k=1}^{N} \sum_{a,b=0}^{n} \gamma_{jk}^{ab}(u, \partial u) \partial_a \partial_b u_k + G_j(u, \partial u), \quad 1 \le j \le N, \tag{1.3}$$

where $G_j = G_j(\lambda, \lambda')$ and $\gamma_{jk}^{ab} = \gamma_{jk}^{ab}(\lambda, \lambda')$ are smooth functions satisfying

$$G_j(0,0) = (\nabla_{\lambda, \lambda'} G_j)(0,0) = 0$$

and $\gamma_{jk}^{ab}(0,0) = 0$.

As far as we consider classical solutions, the system (1.1) can be always reduced to a quasilinear system at the cost of the size N of the system [2] (see [20]). In order to apply the classical energy method for strictly hyperbolic equations, we assume

[1] In some literatures, one writes "semilinear wave equations" for wave equations with nonlinearity of the form $F = F(u)$, such as $F(u) = |u|^{p-1}u$.

[2] We only have to consider an extended system for $(u, \partial u)$.