Appendix C

The proof of Theorem 11.6

In what follows, we put $(a)_+ = \max\{a, 0\}$ for $a \in \mathbb{R}$. First we note that (5.15) for $n = 3$ can be written as

$$\nabla_x \phi(x) = \big(\partial_r \phi(x)\big)\omega - \frac{1}{r}\omega \wedge \widetilde{\Omega}\phi(x) \tag{C.1}$$

for $\phi = \phi(x) \in C^1(\mathbb{R}^3)$, where $r = |x|$, $\omega = |x|^{-1}x$, $\widetilde{\Omega} = (\Omega_{23}, \Omega_{31}, \Omega_{12})$, and $\wedge$ denotes the external product in $\mathbb{R}^3$. Without loss of generality, we may assume $c = 1$ in the proof of Theorem 11.6. Let u be a solution to

$$\begin{aligned} &\Box u = \Phi(t, x), && (t, x) \in (0, T) \times \mathbb{R}^3, \\ &u(0, x) = (\partial_t u)(0, x) = 0, && x \in \mathbb{R}^3. \end{aligned}$$

Given a set $\boldsymbol{c} = \{c_0, c_1, \ldots, c_M\}$ of non-negative numbers, $\mu \geq 0$, $\nu \geq 0$, and $s = 0, 1$, we define

$$\mathcal{Q}_{\boldsymbol{c},\mu,\nu,s}(t) = \sup_{(\tau,y)\in[0,t]\times\mathbb{R}^3} |y|\langle \tau + |y|\rangle^{\mu} \mathcal{W}_{\boldsymbol{c},-}(\tau, |y|)^{\nu} \sum_{|\alpha|+|\beta|\leq s} |(\Omega^\alpha \partial^\beta \Phi)(\tau, y)|.$$

We regard $\mathbb{R}^3$-vectors as column vectors. Let O be an orthogonal matrix, and we put $u_{\mathsf{O}}(t, x) = u(t, \mathsf{O}x)$ and $\Phi_{\mathsf{O}}(t, x) = \Phi(t, \mathsf{O}x)$. Then we have $\Box u_{\mathsf{O}} = \Phi_{\mathsf{O}}$ with $u_{\mathsf{O}}(0, x) = (\partial_t u_{\mathsf{O}})(0, x) = 0$. We get

$$|\partial u_{\mathsf{O}}(t, x)| = |(\partial u)(t, \mathsf{O}x)|, \quad \sum_{|\alpha|+|\beta|\leq 1} |\Omega^\alpha \partial^\beta \Phi_{\mathsf{O}}(t, x)|^2 = \sum_{|\alpha|+|\beta|\leq 1} |(\Omega^\alpha \partial^\beta \Phi)(t, \mathsf{O}x)|^2.$$

Therefore, it suffices to prove Theorem 11.6 for $x = (0, 0, r)$ with $r \geq 0$.

For $0 \leq a \leq 3$, let v_a be a solution to $\Box v_a = (\partial_a \Phi)$ with initial data $v_a(0, x) = (\partial_t v_a)(0, x) = 0$. Also, let w be a solution to $\Box w = 0$ with initial data $w(0, x) = 0$ and $(\partial_t w)(0, x) = \Phi(0, x)$. Then we have

$$\partial_a u(t, x) = v_a(t, x) + \delta_{0a} w(t, x),$$

where δ_{0a} is the Kronecker delta.

Lemma C.1. *Let κ, μ, $\nu > 0$ and $\theta \geq 0$. If $\mu + \nu \geq \kappa + 1$, then we have*

$$\langle t + r\rangle^{1-\theta} \langle t - r\rangle^{\kappa} |w(t, x)| \leq C\mathcal{Q}_{\boldsymbol{c},\mu-\theta,\nu,0}(t).$$

Proof. Since we have

$$\sup_{y\in\mathbb{R}^3} |y|\langle y\rangle^{\kappa+1-\theta} |\Phi(0, y)| \leq \mathcal{Q}_{\boldsymbol{c},\mu-\theta,\nu,0}(0),$$

Theorem 7.2 implies the desired result. □