Appendix B
Distribution-valued functions

In what follows, let I be an open interval on $\mathbb{R}$.

Proposition B.1. *If $f = f(t) \in C^1\big(I; \mathscr{D}'(\mathbb{R}^n)\big)$, then the derivative df/dt as a $\mathscr{D}'(\mathbb{R}^n)$-valued function coincides with the derivative $\partial_t f$ in the sense of $\mathscr{D}'(I \times \mathbb{R}^n)$.*

Proof. We put

$$\lim_{h\to 0} \frac{f(t+h) - f(t)}{h} = g \text{ in } \mathscr{D}'(\mathbb{R}^n),$$

that is to say,

$$\lim_{h\to 0} \left\langle \frac{f(t+h) - f(t)}{h}, \psi \right\rangle_{\mathscr{D}'(\mathbb{R}^n)} = \langle g, \psi \rangle_{\mathscr{D}'(\mathbb{R}^n)}, \quad \psi \in C_0^\infty(\mathbb{R}^n).$$

Recall that

$$\langle f, \varphi \rangle_{\mathscr{D}'(I\times\mathbb{R}^n)} = \int_I \langle f(t), \varphi(t) \rangle_{\mathscr{D}'(\mathbb{R}^n)} dt, \quad \varphi \in C_0^\infty(I \times \mathbb{R}^n).$$

For $\varphi \in C_0^\infty(I \times \mathbb{R}^n)$, we can easily show that $\lim_{h\to 0} \varphi(t+h) = \varphi(t)$ in $C_0^\infty(\mathbb{R}^n)$, and

$$\lim_{h\to 0} \frac{\varphi(t+h) - \varphi(t)}{h} = \partial_t \varphi(t) \text{ in } C_0^\infty(\mathbb{R}^n).$$

Therefore, we have

$$\begin{aligned} \frac{d}{dt} \langle f(t), \varphi(t) \rangle_{\mathscr{D}'(\mathbb{R}^n)} &= \lim_{h\to 0} \left\langle \frac{f(t+h) - f(t)}{h}, \varphi(t+h) \right\rangle_{\mathscr{D}'(\mathbb{R}^n)} \\ &\quad + \lim_{h\to 0} \left\langle f(t), \frac{\varphi(t+h) - \varphi(t)}{h} \right\rangle_{\mathscr{D}'(\mathbb{R}^n)} \\ &= \langle g(t), \varphi(t) \rangle_{\mathscr{D}'(\mathbb{R}^n)} + \langle f(t), \partial_t \varphi(t) \rangle_{\mathscr{D}'(\mathbb{R}^n)}, \end{aligned}$$

which leads to

$$\int_I \langle g(t), \varphi(t) \rangle_{\mathscr{D}'(\mathbb{R}^n)} dt + \int_I \langle f(t), \partial_t \varphi(t) \rangle_{\mathscr{D}'(\mathbb{R}^n)} dt = \int_I \frac{d}{dt} \langle f(t), \varphi(t) \rangle_{\mathscr{D}'(\mathbb{R}^n)} dt = 0.$$

Therefore we obtain

$$\begin{aligned} \langle \partial_t f, \varphi \rangle_{\mathscr{D}'(I\times\mathbb{R}^n)} &= -\langle f, \partial_t \varphi \rangle_{\mathscr{D}'(I\times\mathbb{R}^n)} = -\int_I \langle f(t), \partial_t \varphi(t) \rangle_{\mathscr{D}'(\mathbb{R}^n)} dt \\ &= \int_I \langle g(t), \varphi(t) \rangle_{\mathscr{D}'(\mathbb{R}^n)} dt = \langle g, \varphi \rangle_{\mathscr{D}'(I\times\mathbb{R}^n)}. \end{aligned}$$

This completes the proof. □