Appendix A

The Riemann integral in Banach spaces

A *partition* Π of a bounded closed interval $[a,b]$ is a sequence $\{t_0, t_1, \ldots, t_M\}$ of finite numbers satisfying

$$a = t_0 < t_1 < \cdots < t_{M-1} < t_M = b.$$

We define $|\Pi| := \max_{1 \le m \le M} |t_m - t_{m-1}|$. A pair $(\Pi, \boldsymbol{\tau})$ is called a *tagged partition* of $[a,b]$ if $\Pi = \{t_0, t_1, \ldots, t_M\}$ is a partition of $[a,b]$ and $\boldsymbol{\tau} = \{\tau_1, \ldots, \tau_M\}$ is a sequence of finite numbers satisfying $\tau_m \in [t_{m-1}, t_m]$ for $1 \le m \le M$.

Let $\Pi = \{t_0, t_1, \ldots, t_M\}$ and $\Xi = \{s_0, s_1, \ldots, s_L\}$ be two partitions of $[a,b]$. We say that Ξ is a *refinement* of Π if, for all $m = 0, 1, \ldots, M$, there is $l_m \in \{0, 1, \ldots, L\}$ such that $t_m = s_{l_m}$. A tagged partition $(\Xi, \boldsymbol{\sigma})$ is called a *refinement* of a tagged partition $(\Pi, \boldsymbol{\tau})$ if the partition Ξ is a refinement of the partition Π.

Let X be a Banach space over a field $\mathbb{K}$ with $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. Given a bounded X-valued function f on $[a,b]$ and a tagged partition $(\Pi, \boldsymbol{\tau})$ of $[a,b]$ with $\Pi = \{t_0, t_1, \ldots, t_M\}$ and $\boldsymbol{\tau} = \{\tau_1, \ldots, \tau_M\}$, we define the *Riemann sum*

$$\mathcal{S}(\Pi, \boldsymbol{\tau}; f) := \sum_{m=1}^{M} (t_m - t_{m-1}) f(\tau_m).$$

We say that a bounded X-valued function $f = f(t)$ on $[a,b]$ is *Riemann integrable* on $[a,b]$ if there is $F \in X$ such that $\mathcal{S}(\Pi, \boldsymbol{\tau}; f)$ converges to F in X, uniformly with respect to $\boldsymbol{\tau}$, as $|\Pi| \to 0$. This F is called the *Riemann integral* of f on $[a,b]$, and is written as $\int_a^b f(t)dt$.

Lemma A.1. *Let $f \in C\big([a,b]; X\big)$. For any $\varepsilon > 0$, there is a positive constant δ such that*

$$\|\mathcal{S}(\Pi, \boldsymbol{\tau}; f) - \mathcal{S}(\Xi, \boldsymbol{\sigma}; f)\|_X < \varepsilon$$

for any tagged partition $(\Pi, \boldsymbol{\tau})$ with $|\Pi| < \sigma$, and its refinement $(\Xi, \boldsymbol{\sigma})$.

Proof. Let ε be a positive number. Since f is uniformly continuous on $[a,b]$, there is a positive constant δ such that $\|f(t) - f(s)\|_X < \varepsilon/(b-a)$ for any $t, s \in [a,b]$ with $|t - s| < \delta$.

We write $\Pi = \{t_0, t_1, \ldots, t_M\}$, $\boldsymbol{\tau} = \{\tau_1, \ldots, \tau_M\}$, $\Xi = \{s_0, s_1, \ldots, s_L\}$, and $\boldsymbol{\sigma} = \{\sigma_1, \ldots, \sigma_L\}$. Let $|\Pi| < \delta$, and Ξ be a refinement of Π. Suppose that $t_m = s_{l_m}$ (note that $l_0 = 0$ and $l_M = L$). Then we have $t_m - t_{m-1} = \sum_{l=l_{m-1}+1}^{l_m} (s_l - s_{l-1})$, and $|\sigma_l - \tau_m| \le t_m - t_{m-1} < \delta$ for $l_{m-1} + 1 \le l \le l_m$. Therefore, we get

$$\|\mathcal{S}(\Pi, \boldsymbol{\tau}; f) - \mathcal{S}(\Xi, \boldsymbol{\sigma}; f)\|_X \le \sum_{m=1}^{M} \sum_{l=l_{m-1}+1}^{l_m} (s_l - s_{l-1}) \|f(\tau_m) - f(\sigma_l)\|_X < \varepsilon.$$