Chapter 7

Some supplementary topics

In this chapter, we treat some related results and supplementary topics.

In Section 7.1, we discuss Alexeev's criterion for Serre's S_3 condition (see [Ale3]) with slight generalizations. Note that Alexeev's criterion is a clever application of our new torsion-free theorem (see Theorem 5.6.2 (i) or Theorem 3.16.3 (i)). Although we have already obtained various related results and several generalizations (see, for example, [AH], [Ko12], [Kv6], and so on), we only treat Alexeev's criterion here. Note that log canonical singularities are not necessarily Cohen–Macaulay. In Section 7.2, we collect some basic properties of cone singularities for the reader's convenience. The results in Section 7.2 are useful when we construct various examples. We have already used them several times in this book. In Section 7.3, we give some examples of threefolds. They show that we need flips even when we run the minimal model program for a smooth projective threefold with the unique smooth projective minimal model. This means that we necessarily have singular varieties in the intermediate step of the above minimal model program. In Section 7.4, we describe an explicit example of threefold toric log flips. It may help us understand the proof of the special termination theorem in [F13]. In Section 7.5, we explicitly construct a three-dimensional non-$\mathbb{Q}$-factorial canonical Gorenstein toric flip. It may help us understand the non-$\mathbb{Q}$-factorial minimal model program explained in Section 4.9. In this example, the flipped variety is smooth and the Picard number increases by a flip. In Sections 7.3, 7.4, and 7.5, we use the toric geometry to describe flips. For the combinatorial descriptions of the minimal model program for toric varieties, we recommend the reader to see [R2] and [Mak, Chapter 14].

7.1 Alexeev's criterion for S_3 condition

In this section, we explain Alexeev's criterion for Serre's S_3 condition (see Theorem 7.1.1). It is a clever application of Theorem 5.6.2 (i) (see also Theorem 3.16.3 (i)). In general, log canonical singularities are not Cohen–Macaulay. So, the results in this section will be useful for the study of log canonical pairs.

Theorem 7.1.1 (cf. [Ale3, Lemma 3.2]). *Let (X, B) be a log canonical pair with $\dim X = n \geq 3$ and let $P \in X$ be a scheme theoretic point such that $\dim \overline{\{P\}} \leq n-3$. Assume that $\overline{\{P\}}$ is not a log canonical center of (X, B). Then the local ring $\mathcal{O}_{X,P}$ satisfies Serre's S_3 condition.*

We slightly changed the original formulation. The following proof is essentially the same as Alexeev's. We use local cohomologies to calculate depths.