Chapter 6

Fundamental theorems for quasi-log schemes

This chapter is the main part of this book. In this chapter, we introduce the notion of quasi-log schemes and establish the fundamental theorems for quasi-log schemes. We will see that the cone and contraction theorem holds for quasi-log schemes.

Section 6.1 is an overview of the main results of this chapter. In Section 6.2, we introduce the notion of quasi-log schemes. Note that our treatment is slightly different from Ambro's original theory of quasi-log varieties (see [Am1]). In Section 6.3, which is the main part of this chapter, we discuss various basic properties, for example, adjunction and vanishing theorems, of quasi-log schemes. In Section 6.4, we show that a normal pair has a natural quasi-log structure. By this fact, we can apply the theory of quasi-log schemes to normal pairs. We also treat toric polyhedra as examples of quasi-log schemes. Section 6.5 is devoted to the proof of the basepoint-free theorem for quasi-log schemes. It is a powerful generalization of the Kawamata–Shokurov basepoint-free theorem for kawamata log terminal pairs. In Section 6.6, we prove the rationality theorem for quasi-log schemes. In Section 6.7, we discuss the cone and contraction theorem for quasi-log schemes. Thus we establish the fundamental theorems for quasi-log schemes. In Section 6.8, we discuss some properties of quasi-log Fano schemes and related topics. Section 6.9 is devoted to the proof of the basepoint-free theorem of Reid–Fukuda type for quasi-log schemes. Here, we prove it under some extra assumptions. For the details of the basepoint-free theorem of Reid–Fukuda type for quasi-log schemes, see the author's paper [F41].

6.1 Overview

In this chapter, we establish the fundamental theorems for quasi-log schemes. This means that we prove adjunction (see Theorem 6.3.5 (i)), various Kodaira type vanishing theorems (see Theorem 6.3.5 (ii) and Theorem 6.3.8), basepoint-free theorem (see Theorem 6.5.1), rationality theorem (see Theorem 6.6.1), cone and contraction theorem (see Theorem 6.7.4), and so on, for quasi-log schemes after we introduce the notion of quasi-log schemes. Note that our formulation of the theory of quasi-log schemes is slightly different from Ambro's original one in [Am1].

In this book, we adopt the following definition of quasi-log schemes.

Definition 6.1.1 (Quasi-log schemes, see Definition 6.2.2)**.** A *quasi-log scheme* is a scheme X endowed with an $\mathbb{R}$-Cartier divisor (or $\mathbb{R}$-line bundle) ω on X, a proper