Chapter 5

Injectivity and vanishing theorems

The main purpose of this chapter is to establish the vanishing and torsion-free theorem for simple normal crossing pairs (see Theorem 5.1.3), which is indispensable for the theory of quasi-log schemes discussed in Chapter 6. Our approach in this chapter is Hodge theoretic. We use the theory of mixed Hodge structures on cohomology with compact support as a main ingredient. We also use some recent results on the partial resolution of singularities for reducible varieties.

In Section 5.1, we explain the main results of this chapter. Our formulation is natural and useful from the minimal model theoretic viewpoint, although it may look unduly technical and artificial. More precisely, we formulate various vanishing and torsion-free theorems in order to make them suitable for the theory of quasi-log schemes in Chapter 6. In Section 5.2, we review the notion of $\mathbb{Q}$-divisors and $\mathbb{R}$-divisors again for the reader's convenience. This is because we have to treat reducible varieties from this chapter. In Section 5.3, we quickly review Du Bois complexes and Du Bois singularities. We use them in Section 5.4. In Section 5.4, we prove the Hodge theoretic injectivity theorem. It is a correct and powerful generalization of Kollár's injectivity theorem from the Hodge theoretic viewpoint. Our proof depends on the theory of mixed Hodge structures on cohomology with compact support. In Section 5.5, we generalize the Hodge theoretic injectivity theorem for the relative setting. The relative version of the Hodge theoretic injectivity theorem gives a new injectivity theorem and then simplifies the proof of the torsion-free theorem for simple normal crossing pairs in Section 5.6. Section 5.6 is devoted to the proof of the injectivity, vanishing, and torsion-free theorems for simple normal crossing pairs. In Section 5.7, we treat the vanishing theorem of Reid–Fukuda type for embedded simple normal crossing pairs. As an application, we obtain an ultimate generalization of the Kawamata–Viehweg vanishing theorem for log canonical pairs. In Section 5.8, we treat embedded normal crossing pairs. Note that the results in Section 5.8 are not necessary for the theory of quasi-log schemes discussed in Chapter 6. So the reader can skip Section 5.8. Section 5.9 contains many nontrivial examples, which help us understand the results discussed in this chapter. We see that a naive generalization of the Kawamata–Viehweg vanishing theorem does not necessarily hold for varieties with log canonical singularities.

5.1 Main results

In this chapter, we prove the following theorems. Theorem 5.1.1 is a complete generalization of Lemma 3.1.1.