Chapter 4

Minimal model program

In this chapter, we discuss the minimal model program. Although we explain the recent developments of the minimal model program mainly due to Birkar–Cascini–Hacon–M^{c}Kernan in Section 4.4, we do not discuss the proof of the main results of [BCHM]. For the details of [BCHM], see [BCHM], [HaKo, Part II], [HaMc1], [HaMc2], and so on. The papers [Dr], [F25], and [Ka7] are survey articles on [BCHM]. For slightly different approaches, see [BirPa], [CoLa], [CaL], [P], and so on. In this book, we mainly discuss the topics of the minimal model program which are not directly related to [BCHM].

In Sections 4.1, 4.2, and 4.3, we quickly review the basic results on the minimal model program, X-method, and so on. The X-method, which was introduced by Kawamata, is a powerful method for proving some fundamental results in the minimal model program. Unfortunately, the X-method does not work well for log canonical pairs although it is very powerful for kawamata log terminal pairs. Section 4.4 is devoted to the explanation on [BCHM] and some related results and examples. We give a relatively simple proof of the existence of dlt blow-ups as an application of [BCHM]. In Section 4.5, we discuss the fundamental theorems for normal pairs (see [F28]) and various examples of the Kleiman–Mori cone. The results in Section 4.5 are sufficient for the minimal model program for log canonical pairs. Therefore, we strongly recommend the reader to see [F28] if he is only interested in the minimal model program for log canonical pairs. In Section 4.6 and Section 4.7, we prove that Shokurov polytope is a polytope, which is useful and indispensable for the minimal model program for $\mathbb{R}$-divisors. In Section 4.8, we discuss the minimal model program for log canonical pairs and various conjectures. In Section 4.9, we explain the minimal model program for (not necessarily $\mathbb{Q}$-factorial) log canonical pairs. It is the most general minimal model program in the usual sense. In Section 4.10, we review the minimal model theory for singular surfaces following [F29]. Our minimal model theory for log surfaces is much more general than the traditional one. In Section 4.11, we discuss the generalized abundance conjecture. We translate the good minimal model conjecture, which is geometric, into a numerical condition, that is, the generalized abundance conjecture. In Section 4.12, we treat Iitaka's conjectures on κ and $\overline{\kappa}$ and explain the relationship with the generalized abundance conjecture. In Section 4.13, we quickly explain the author's recent result on semi-log canonical pairs without proof. Roughly speaking, every quasi-projective semi-log canonical pairs has a natural quasi-log canonical structure.