Chapter 3
Classical vanishing theorems and some applications

In this chapter, we discuss various classical vanishing theorems, for example, the Kodaira vanishing theorem, the Kawamata–Viehweg vanishing theorem, the Fujita vanishing theorem, and so on. They play crucial roles for the study of higher-dimensional algebraic varieties. We also treat some applications. Although this chapter contains some new arguments and several new results, almost all results are standard or known to the experts. Of course, our choice of topics is biased and reflects the author's personal taste.

In Section 3.1, we give a proof of the Kodaira vanishing theorem for smooth projective varieties based on the theory of mixed Hodge structures on cohomology with compact support. It is slightly different from the usual one but suits for our framework discussed in Chapter 5. In Sections 3.2, 3.3, and 3.4, we prove the Kawamata–Viehweg vanishing theorem, the Viehweg vanishing theorem, and the Nadel vanishing theorem, respectively. They are generalizations of the Kodaira vanishing theorem. In Section 3.5, we prove the Miyaoka vanishing theorem as an application of the Kawamata–Viehweg–Nadel vanishing theorem. Note that the Miyaoka vanishing theorem is the first vanishing theorem for the integral part of $\mathbb{Q}$-divisors. Section 3.6 is a quick review of Kollár's injectivity, torsion-free, and vanishing theorems without proof. We will prove complete generalizations in Chapter 5. In Section 3.7, we treat Enoki's injectivity theorem, which is a complex analytic counterpart of Kollár's injectivity theorem. In Sections 3.8 and 3.9, we discuss Fujita's vanishing theorem and its applications. In Section 3.10, we quickly review Tanaka's vanishing theorems without proof. They are relatively new and are Kodaira type vanishing theorems in positive characteristic. In Section 3.11, we prove Ambro's vanishing theorem as an application of the argument in Section 3.1. In Section 3.12, we discuss Kovács's characterization of rational singularities. Kovács's result and its proof are very useful. In Section 3.13, we prove some basic properties of divisorial log terminal pairs. In particular, we show that every divisorial log terminal pair has only rational singularities as an application of Kovács's characterization of rational singularities. In Example 3.13.9, we explicitly construct a flopping contraction from a divisorial log terminal threefold whose flopping curve is an elliptic curve and flopped pair is not divisorial log terminal. This example helps us understand some subtleties of divisorial log terminal pairs. Section 3.14 is devoted to the Elkik–Fujita vanishing theorem and its application. We give a simplified proof of the Elkik–Fujita vanishing theorem due to Chih-Chi Chou. In Section 3.15, we explain the method of two spectral sequences of local cohomology groups. Section 3.16 is an introduction to our new vanishing theorems. We will