Chapter 2
Preliminaries

In this chapter, we collect the basic definitions of the minimal model program for the reader's convenience.

In Section 2.1, we recall some basic definitions and properties of $\mathbb{Q}$-divisors and $\mathbb{R}$-divisors. The use of $\mathbb{R}$-divisors is indispensable for the recent developments of the minimal model program. Moreover, we have to treat $\mathbb{R}$-divisors on reducible non-normal varieties in this book. In Section 2.2, we recall the basic definitions and properties of the Kleiman–Mori cone and explain some interesting examples. Note that Kleiman's famous ampleness criterion does not always hold for complete *non-projective* singular algebraic varieties. In Section 2.3, we discuss discrepancy coefficients, singularities of pairs, negativity lemmas, and so on. They are very important in the minimal model theory. In Section 2.4, we quickly recall the Iitaka dimension, the numerical Iitaka dimension, movable divisors, pseudo-effective divisors, Nakayama's numerical dimension, and so on. In Section 2.5, we discuss the invariant Iitaka dimension for $\mathbb{R}$-divisors due to Sung Rak Choi.

2.1 Divisors, $\mathbb{Q}$-divisors, and $\mathbb{R}$-divisors

Let us start with the definition of simple normal crossing divisors and normal crossing divisors.

Definition 2.1.1 (Simple normal crossing divisors and normal crossing divisors)**.** Let X be a smooth algebraic variety. A reduced effective Cartier divisor D on X is said to be a *simple normal crossing divisor* (resp. *normal crossing divisor*) if for each closed point p of X, a local defining equation f of D at p can be written as

$$f = z_1 \cdots z_{j_p}$$

in $\mathcal{O}_{X,p}$ (resp. $\widehat{\mathcal{O}}_{X,p}$), where $\{z_1, \cdots, z_{j_p}\}$ is a part of a regular system of parameters.

Note that the notion of $\mathbb{Q}$-factoriality plays important roles in the minimal model program.

Definition 2.1.2 ($\mathbb{Q}$-factoriality)**.** A normal variety X is said to be $\mathbb{Q}$-*factorial* if every prime divisor D on X is $\mathbb{Q}$-Cartier, that is, some non-zero multiple of D is Cartier.

Example 2.1.3 shows that the notion of $\mathbb{Q}$-factoriality is very subtle.

Example 2.1.3 (cf. [Ka5])**.** We consider

$$X = \{(x, y, z, w) \in \mathbb{C}^4 \mid xy + zw + z^3 + w^3 = 0\}.$$